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OPTIMAL DECENTRALIZED LOAD FREQUENCY CONTROL FOR POWER SYSTEM: A MEAN-FIELD TEAM APPROACH

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DEDICATION

À ma famille,

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I am sincerely thankful to my supervisor Professor Roland P.Malhamé at Polytechnique Montréal for his continuous guidance and supervision throughout the research work.

I would like to thank him for introducing me to this very challenging and interesting topic of load frequency control and for providing a lot of technical guidance for the thesis.

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RÉSUMÉ

Le problème de réglage fréquence-puissance (RFP) dans les réseaux électriques connaît un regain récent d'intérêt vu la pénétration de plus en plus importante dans ces réseaux de sources d'énergie renouvelable solaire ou éolienne, c'est à dire avec caractéristiques d'intermittence. En effet, la fréquence est un signal dont le comportement est sensible à tout déséquilibre entre génération et demande d'électricité, et son maintien dans un voisinage serré de sa valeur nominale (60 Hz en Amérique du Nord), est essentiel pour la stabilité du réseau. Le RFP vise à contrôler la puissance de sortie des générateurs en réponse aux changements de fréquence (dans le cas d'une zone unique) ainsi qu'à ceux des échanges d'énergie par rapport à leur valeur programmée dans les lignes de raccordement (dans le cas de zones multiples). Les techniques actuelles de RFP présentent un mélange de caractéristiques de centralisation et de décentralisation. Dans ce mémoire, nous souhaitons revisiter les algorithmes de RFP à la lumière des derniers développements de la théorie des équipes et jeux à champ moyens (mean field teams and mean field games), en exploitant le fait que le signal de fréquence global utilisé pour coordonner les générateurs est en réalité une moyenne pondérée des fréquences locales à un grand nombre de générateurs. Nous explorerons ainsi dans ce mémoire des approches de commande intégrale-proportionnelle avec structure de coût quadratique pour le RFP.

Le problème de commande est formulé comme un problème d'équipe linéaire quadratique selon la structure d'information champ-moyen partagé, c'est-à-dire que chaque générateur observe son propre état (incluant 3 variables internes supposées mesurables) ainsi que le champ moyen consistant en une moyenne des états de tous les générateurs. La commande décentralisée correspond à la solution du problème d'équipe à champ moyen. Cette dernière est obtenue en résolvant 2 équations de Riccati dans le cas d'une zone isolée, l'une associée au générateur local et l'autre associée au champ moyen (ces équations deviennent des équations de Riccati couplées dans le cas d'un problème à 2 zones).

Le mémoire est composé de deux parties dédiées respectivement à l'analyse de la commande pour une zone isolée, et celle associée à deux zones interconnectées.

Dans la première partie, nous introduisons la théorie de l'équipe dans le contrôle RFP à zone unique. Il s'agit de problèmes de décision multi-agents dans lesquels tous les agents (générateurs individuels) partagent un coût commun [Mahajan et al., 2012]. L'approche de commande actuelle utilise une contrainte de taux de génération unique [Tan, 2010] pour contrôler le comportement de tous les agents/machines et la répartition de la charge se fait en se référant à la taille de la machine, ce qui est simple mais apriori un peu trop grossier.

En appliquant la théorie de l'équipe, chaque individu pourrait avoir un contrôleur local qui réagit en fonction de sa propre situation actuelle et un contrôleur global qui cherche a mener la moyenne vers l'objectif désiré. Nous prenons en compte à la fois les cas homogène et non homogène, où dans le cas non homogène nous supposons qu'il existe 3 types de générateurs différents, représentant des générateurs de taille petite, moyenne et grande. Nous développons d'abord le modèle dynamique pour chaque générateur, puis formulons la fonction coût commune aux agents et étudions et identifions la stratégie optimale.

Dans la deuxième partie, nous étudions un réseau électrique de deux zones interconnectées. Plus complexe que le cas de la zone unique, le système de 2 zones est affecté non seulement par les déviations des fréquences locales aux zones, mais également par les déviations de puissance dans leurs interconnexions mutuelles par rapport aux niveaux programmés de façon contractuelle. Ici la théorie de l'équipe est une approche inadéquate car elle ne peut traiter que du cas où les fonctions coûts sont partagées par les machines des deux zones; or celles-ci sont en général gérées par des opérateurs indépendants et qui seront plus soucieux de la performance de leur zone propre. Toutefois, nous proposons d'abord la solution lorsque les deux zones agissent comme une équipe dans un contexte de champ moyen. Il est cependant plus naturel de traiter le problème par une formulation de théorie des jeux avec un comportement d'équipe dans chaque zone et de la concurrence entre les deux zones. Nous formulons alors dans un deuxième temps un jeu à champ moyen entre les deux zones, mais tel que les agents dans chaque zone agissent comme équipe. Nous obtenons les équations correspondant à un équilibre de Nash. L'intérêt de cette approche est de permettre à chaque zone de choisir sa propre stratégie dans le système à 2 zones, ce qui signifie qu'elle peut renforcer ou diminuer le transfert de puissance d'une zone à l'autre en cas de besoin en ajustant sa propre matrice de pondération des coûts. Nous introduisons également un algorithme établi dans la littérature pour résoudre le système d'équations de Riccati couplées pour le cas de la formulation jeux à champ moyen.

ABSTRACT

Load frequency control or LFC is a fundamental mechanism for maintaining the stability of electric power systems. It aims at controlling the power output of generators in response to either changes in frequency (in a single area case) or in response to both changes in frequency and tie-line power interchange (in multi- area cases). Indeed frequency is a ubiquitous signal in power systems and its excursions away from its nominal value are indicative of imbalances between generation and load. Interest in LFC has come back to the fore in view of the challenges raised by increasing levels of penetration of renewable intermittent sources (wind and solar energy). This situation creates frequent and important mismatches between system generation and system load, and thus create the need for more effective LFC schemes. The current set up is based on estimating a single integral control based power mismatch variable and redistributing a share of the correspondingly needed generation increase or decrease among units according to their power rating [Tan, 2010]. While this has proved to be a robust and algorithmically simple scheme, it is a rather rough approach, as it tends to ignore the particular current state of each generator when provided with a new set point. In order to allow more flexible and less aggressive control to each individual generator, normally only represented as a single aggregate unit, novel decentralized linear quadratic-proportional integral control methods for load frequency control respectively based on so-called mean field team theory (for single and two area systems) and mean field games (for two area systems) are discussed in this thesis.

The control problem is formulated as a linear quadratic (LQ) team problem under mean-field sharing (MFS) information structure, i.e., each generator observes its own state (3 state variables) and the mean field, that is in this context the average state of all generators if they are all identical, or the vector of class specific mean states in a non homogeneous multi-class situation. Also, following a team solution scheme developed in [Arabneydi and Mahajan, 2016], a separate mean field control term is a feedback on the vector of mean class specific individual states. The overall result is a decentralized control policy with coordination by the mean field term. The optimal solution is obtained by solving 2 Riccati equations, one for the local generator and another one associated with the mean field (this becomes instead a system of coupled Riccati equations in the subsequent mean field game game solution of the 2 area problem), for the full observation model.

This thesis consists of two parts wherein each part introduces a new control method:

In the first part, we introduce team theory into single area LFC control. The team optimal

control of decentralized systems investigates multi-agent decision problems in which all agents share a common objective [Mahajan et al., 2012].

By applying team theory, each individual could have a local controller that adapts its effort to the local current state and a global controller which insures that the average state follow the desired target. We consider both homogeneous and non homogeneous cases, where in the non homogeneous case we assume at most 3 different types of generators, coarsely representing the classes of small, medium and large generators. We first develop the dynamic model of each generator, then formulate the common cost objective and identify the optimal strategy.

In the second part of the thesis, we study 2 area power systems. Two areas systems are more complex than single area ones in that each area is affected not only by area specific frequency deviations from nominal value, but also by inter area tie line power flow deviations from contract based values. A team theory solution is first considered, although it fall short of the requirements since, because it is globally optimal, it does not reflect the fact that in reality, the two areas may in general be operated by different companies which will be self-interested. Thus subsequently, we analyze a formulation whereby machines within the same area act as one team, while they are non cooperative gamelike between two areas. Coupled Riccati equations characterizing a Nash equilibrium solution are developed. The novelty of this approach relative to the complete two area team one is that each area is now allowed to weigh differently the frequency and tie line related states in their cost function, thus trying to achieve trade-offs between selfish area oriented priorities, and the possibility of each area helping the other one in case of need through the tie line power exchange term.

Finally we implement an algorithm found in the existing literature, for solving the coupled Riccati equations system .

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LIST OF SYMBOLS AND ABBREVIATIONS

LQ	Linear Quadratic
MF	Mean Field
MFS-IS	Mean Field Sharing Information Structure
ACE	area control error
AGC	Automatic generation Control

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CHAPTER 1 INTRODUCTION

1.1 Motivation

Load frequency control or LFC is a fundamental mechanism for maintaining the stability of electric power systems. It aims at controlling the power output of generators in response to either changes in frequency (in single area case) or in response to both changes in frequency and tie-line power interchange (in multi- area cases). Indeed frequency is a ubiquitous signal in power systems and its excursions away from its nominal value are indicative of imbalances between generation and load. Interest in LFC has come back to the fore in view of the challenges raised by increasing levels of penetration of renewable intermittent sources (wind and solar energy). This situation creates frequent and important mismatches between system generation and system load, and thus create the need for more effective LFC schemes.

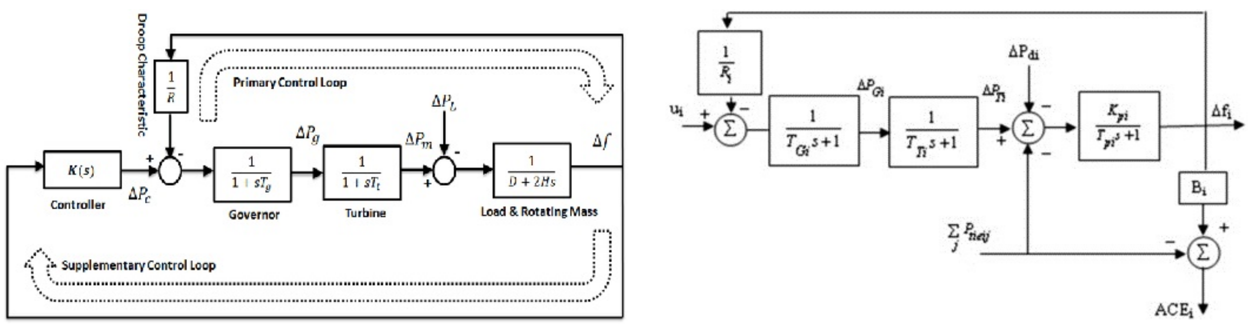

Figure 1.1 Current areawise area load frequency control with feedback on frequency deviation(left) and with frequency deviation and tie line power interchange(right)

The current set up, shown in Figure 1.1, is based on estimating a single integral control based power mismatch variable and redistributing a share of the correspondingly needed generation increase or decrease among units according to their power rating [Tan, 2010]. While this has proved to be a robust and algorithmically simple scheme, it is a rather rough approach, as it tends to ignore the particular current state of each generator when provided with a new set point, thus each generator may deviate from its nominal value even when the areawise average remain unchanged.

In order to allow more detailed and individualized control to each individual generator, normally only represented by a single aggregate unit, new generator set up , which separate cooperative generators into multi individual 'virtual' macromachine will be presented, and a novel decentralized linear quadratic-proportional integral control methods for load frequency control respectively based on so-called mean field team theory (for single and two area systems) and non cooperative games between mean field teams (for two area systems) will be discussed in this thesis.

1.2 Description of the Project Studied

In this section, we briefly introduce mean-field teams and game theory in the context of LFC problems and describe the main challenges, ideas, and results.

Background: Linear Quadratic Mean-field Team Theory

Presented in detail in Appendix A, linear quadratic mean-field team theory is a team optimal control of decentralized systems with linear dynamics and quadratic costs that consist of multiple sub-populations, where the dynamics and costs are coupled across agents through the mean-field (or empirical mean) of the states and actions.

An non-classical information structure, *mean-field sharing* information structure, is investigated where all agents observe their local state and the mean-field of all sub-populations.

In [Arabneydi, 2016], it is shown that this linear control strategies are unique and optimal and the corresponding gain could be determined in a decentralized manner, where each agent solves 2 independent Riccati equations, one for itself and one for the mean field. The dimensions of these Riccati equations are independent of the size of sub-populations.

This recent control theory allows us to control each single macromachine in the generator system with 2 controllers: one local controller which contains feedback on all internal variables of a single machine, and a global controller which contains feedback on the mean-field variables , thus achieving overall optimality without resorting to unduly complex controllers.

Part 1: Single Area LFC Problem

In Part 1, we present the single area LFC mean field team problem first under the special condition of homogeneous machines; subsequently, we move to an analysis of the more general non homogeneous case. Our purpose here is to use mean field team theory [Arabneydi and Mahajan, 2016] to reinterpret in a more rigorously founded way some of the current "heuristically based" control practices, yet still achieve a decentralized control scheme with coordination signals through a mean field (here understood as the average machine state) component. we shall work under the following constraints:

(i) Agents must have similar but individualized linear models which contains the particular current state of each generator (unlike the current modelling framework which uses a single large "aggregate machine"). (ii) Control actions include two components: one locally decided by each device, one imposed by a coordinating mean field controller and help preserve global optimality. (iii) Unpredicted load/generation changes are considered as the disturbance to users load profile relative to the uncontrolled situation, and the overall controller should be sufficiently robust or adaptive to mitigate their effect on system stability.

Part 2: 2 Area LFC Problem

Similarly to Part 1, in Part 2 mean-field team theory is first implemented, however this time on a two area system. Two area systems are different from single area systems in that, besides the objective of maintaining each area frequency at its nominal value, there is also concern with maintaining inter area tie line power flows at their scheduled levels. Thus the global cost function is modified accordingly, and following the current power system control approaches, we introduce the so-called area control error signals, ACE [Kundur et al., 1994], as states in the system.

The mean field team formulation leads to a globally optimal solution which is of interest in its own right. However, as mentioned earlier, the two area situation may require a different treatment as the areas in question may be operated by different companies, which in general will be more concerned with the dynamic behavior of the states internal to their area. Thus, it is more appropriate to analyse the situation as a game between two mean field teams. As a result, the objectives in the latter sections of part 2 are:

(i) applying non cooperative game theory to the two areas problem where each individually acts as a mean field team. (ii) introducing the popular concept ACE into the cost functions of each area. (iii) comparing the behaviour of the mixed team-game solution to that of the mean field team solution.

1.3 Organization of the Thesis

The thesis is organized as follows.

Chapter 2 presents a literature review of existing load frequency control approaches as well as a mean field team application in the power system area.

Chapter 3 develops a new model of generator for a single area system, which is also best suited for a mean field team view of the control problem. An adequate quadratic cost function is formulated and the optimal team solution structure is characterized for both homogeneous and non homogeneous cases. Chapter 4 is dedicated to the same issues but in the case of a two area system. Also a game theoretic analysis of the interacting two areas is carried out.

Chapter 5 discusses the conclusion and possible extensions to the current study.

CHAPTER 2 LITERATURE REVIEW

LFC has been a well developed area due to the early need for efficient frequency control methods for large-scale power systems. As a result, many papers can be found dealing with aspects such as automatic generation control, its coupling with machine excitation systems control, with robustness objectives against parameter uncertainties, plant/model mismatch and external load changes [Dubey and Bondriya, 2016]. A well designed control method should be able to be resilient in the presence of load and generation disturbances, so as to keep both voltage and frequency within tolerance limits. Various method has been proposed in recent years, such as optimal control, variable structure control and adaptive/self-tuning/fuzzy control methods.

Among these methods, classical optimization theory was used to find the "best" value of parameters [Kundur et al., 1994], with a typical PI/PID tuning approach [Saadat, 1999]. The main challenge of these types of controller is the tuning of the integral gain in that it may cause cause large oscillations and instability, and lead to saturation. A compromise must be found between a sufficiently responsive frequency control system while maintaining a low overshoot.

It is the difficulty of reaching such a satisfactory compromise that has lead to early exploration of the application of modern optimal control theory to the LFC problem (see [Fosha and Elgerd, 1970]), where the authors develop a linear quadratic regulator formulation that leads to novel insights although after long discussions, the power industry decided to continue with the current classical control inspired schemes.

Based on PI and PID parameterized control structures, global optimization techniques have been explored such as a decentralized unified PID tuning approximation with a standard second-order plus deadtime (SOPDT) model [Tan, 2009], a new decentralized robust optimal MISO PID controller [Yazdizadeh et al., 2012], Ziegler-Nichols method [Chiha et al., 2012], genetic algorithms (GA)[Bagis, 2007], particle swarm optimization [Selvan et al., 2003], or ant colony optimization [Hsiao et al., 2004]. Riccati equation, $H\infty$, μ-synthesis, potent pole venture, loop shaping, linear matrix inequality (LMI) have also been applied to design the power system controllers [Dubey and Bondriya, 2016].

In the first part of our research, we extend the study of linear quadratic regulator methods to linear quadratic mean-field team theory [Arabneydi and Mahajan, 2014, 2015] in the single area LFC problem.

In particular, we consider team optimal control of decentralized systems with linear dynamics and quadratic costs where the agents are coupled in the dynamics and cost through the mean-field (i.e., the empirical mean) of the states and actions. We revisit the control system of each single generator, which is normally considered as a big single system and controlled by a single Generation rate constraint(GRC) [Tan, 2010]. It is shown that linear control strategies are optimal. The corresponding gains are determined by the solution of K + 1 Riccati equations, where K is the number of agents.

The situation is much more complex in the multi area case. Supplied by hundreds of inter-connected generating units, small or large, an interconnected power system could be highly non-linear and large scale, with a variety of loads and some overlap across areas through highly meshed transmission lines, and a distribution network. Instabilities of different form can arise in this case.

The objective of the LFC in such a multi area situation is to keep both the frequency of every area within limits and to hold tie-line power flows inside some pre-specified tolerances by way of adjusting the MW outputs of the generators with the intention to accommodate fluctuating load needs[Dubey and Bondriya, 2016].

Additional tuning algorithms have been applied in this area using soft computing methods such as artificial neural network (ANN) based approaches [Khuntia and Panda, 2012], fuzzy logic and self-Adaptive modified bat algorithm (SAMBA) [Shoults and Ibarra, 1993],self-tuning Regulators(STR) [Shoults and Ibarra, 1993], to deal with the difficulties in the design because of non-linearity in more than a few segregated add-ons of the controller.

CHAPTER 3 SINGLE AREA LFC

3.1 Problem Formulation

3.1.1 Notation

Table 3.1 Notation for single area LFC

Symbol	Description
K_P	Electric system gain(Hz/p.u.MW)
T_P	Electric system time constant (s)
T_T	Turbine time constant (s)
T_G	Governor time constant (s)
$\Delta f(t)$	Incremental frequency deviation (Hz)
$\Delta P_g(t)$	Incremental change in generator output(p.u.MW)
$\Delta X_g(t)$	Incremental change in governor value position(p.u.MW)
ΔP_d	Load disturbance (p.u.MW)
u	Incremental change in the speed changer position(p.u.MW)

$\mathbb{N}$ and $\mathbb{R}$ denote natural and real numbers, respectively. Given matrices A, B, Q, and R with appropriate dimensions, Feedback(A, B, Q, R) denotes $-R^{-1}B^\intercal M$, where M is a symmetric matrix satisfying the following algebraic Riccati equation:$A^\intercal M + MA - MBR^{-1}B^\intercal M + Q = 0$.

Let $y^i_{ref_k}(t) \in \mathbb{R}$, $u^i_k(t) \in \mathbb{R}$, and $y^i_k(t) \in \mathbb{R}$ denote the tracking reference, control action, and the output of system $i \in \{1, \ldots, n\}$ at time $t \in [0, \infty)$, where $y \in \{\Delta f, \Delta P_g, \Delta X_g\}$, listed in Table 3.1. According to the results in [Arabneydi, 2016] on optimal mean field teams, each system i is controlled by two controllers: a local controller with gains L^i_k, k refers to the index class of the machine, which controls the individualized generator state $\Delta f^i(t)$, $\Delta P^i_g(t)$, $\Delta X^i_g(t)$; and a global controller with gains G, which controls the mean-field, $\int_{\tau=0}^{t}(\Delta \bar{f}\tau)d\tau$, $\Delta \bar{f}(t)$, $\Delta \bar{P}_g(t)$, $\Delta \bar{X}_g(t)$.

In a single area homogeneous case, the index k is suppressed, the local controller $L^i := (L^i_f, L^i_p, L^i_x)$, contains the local feedback on $\Delta f^i(t)$, $\Delta P^i_g(t)$ and $\Delta X^i_g(t)$, respectively; and the global controller $G := (G_I, G_f, G_p, G_x)$ contains the feedback on mean-field $\int_{\tau=0}^{t}(\Delta \bar{f}\tau)d\tau$, $\Delta \bar{f}(t)$, $\Delta \bar{P}_g(t)$, $\Delta \bar{X}_g(t)$, respectively;

In single area non homogeneous case, $k \in \{1, 2, 3\}$ denotes the 3 different types of generators, respectively small, medium and large with $L_k^i := (L_{f_k}^i, L_{p_k}^i, L_{x_k}^i)$ represents the local controller for agent i in group k with feedback on $\Delta f^{i_k}(t)$, $\Delta P_g^{i_k}(t)$ and $\Delta X_g^{i_k}(t)$, respectively;
and the global controller $G_k := (G_{I_k}, K_{f_{1k}}, K_{p_{1k}}, K_{x_{1k}}, K_{f_{2k}}, K_{p_{2k}}, K_{x_{2k}}, K_{f_{3k}}, K_{p_{3k}}, K_{x_{3k}})$ contains the feedback for mean-field $\int_{\tau=0}^{t}(\Delta \bar{f}\tau)d\tau$, $\Delta \bar{f}_1(t)$, $\Delta \bar{P}_{g_1}(t)$, $\Delta \bar{X}_{g_1}(t)$, $\Delta \bar{f}_2(t)$, $\Delta \bar{P}_{g_2}(t)$, $\Delta \bar{X}_{g_2}(t)$, $\Delta \bar{f}_3(t)$, $\Delta \bar{P}_{g_3}(t)$, $\Delta \bar{X}_{g_3}(t)$, respectively;

Assumption 1. *All Reference trajectories are constant.*

3.1.2 Information Structure

In a single area, at time t, with only one machine class in the homogeneous case, and up to 3 machine classes in the non homogeneous case, a generic generator $i \in \{1, \ldots, n_k\}$ of class k, $k = \{1, 2, 3\}$, is assumed to observe its own output vector $y_k^i = [\Delta f_k^i, \Delta P_{g_k}^i, \Delta X_{g_k}^i]^T$, and in general all averages $\bar{y}_l(t) := \frac{1}{n_l}\sum_{j=1}^{n} y_l^j(t)$ (also called *class wide mean fields*), $l = \{1, 2, 3\}$, with y_l^j denoting the output of the j^{th} generator of class l. This information structure is decentralized with a coordinating mean field and we refer to it as *mean-field sharing.* In practice, there are different ways to share the mean-field terms among systems. For example, a central authority can collect each system's output, compute the needed averages, and broadcast them to all machines. Or, systems can run a consensus-type algorithm such as [Xiao and Boyd, 2004] to share the class wide vector of averages among themselves in a distributed manner. The control block diagram of the overall machine system under mean-field sharing information structure of the single area with homogeneous machines denoted A_1^H is depicted in Figure 3.4 (further below), while that of the single area with multi class machines denoted A_1^{NH} is represented in Figure 3.6 (further below).

3.1.3 The Single Area Optimization Problems: Problem 1 and Problem 2

Optimization Problem 1: Homogeneous Case

We are interested in finding local gains for each agent $L := (L^1, \ldots, L^n)$ and global gain for mean-field G such that the average of output frequencies $\Delta \bar{f}(t)$ is close to a tracking reference $\Delta \bar{f}_{ref}(t) \in \mathbb{R}$ while the distance between output of machine i, $\Delta f^i(t)$ and its tracking reference $\Delta f_{ref}^i(t)$ is minimum. To this end, we define the following performance index for the team optimal solution for single area homogeneous case where a mean frequency time integral related cost has been added to robustify the behaviour against piecewise constant load or generation disturbances:

$$
\begin{aligned}
J_1^H(L,G) := \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\Bigg[\frac{1}{n}\sum_{i=1}^{n}\Big[q_f\left(\Delta f^i(t)-\Delta f^i_{ref}(t)\right)^2+Ru^i(t)^2\Big] \\
+p_f\Big(\Delta\bar{f}(t)-\Delta\bar{f}_{ref}(t)\Big)^2+\tilde{p}\Big(\int_{\tau=0}^{t}(\Delta\bar{f}(\tau)-\Delta\bar{f}_{ref}(\tau))d\tau\Big)^2 dt\Bigg],
\end{aligned}
\tag{3.1}
$$

Problem 1. *The objective is to minimize the cost function* J_1^H *under mean-field sharing information structure such that:*

$$
J_1^{H^*} := J_1^H(L^*,G^*) \leq J_1^H(L,G). \tag{3.2}
$$

L,G *should be independent of the choice of reference trajectories.*

Optimization Problem 2: Non-homogeneous Case

We are interested in finding gains $L=(L_1,L_2,L_3)$ where $L_k := (L_k^1,\dots,L_k^{n_k})$ is the local controller for generator class $k=\{1,2,3\}$, and G such that the average of all outputs $\Delta\bar{f}(t)$ is close to a tracking reference $\Delta f_{ref}(t)\in\mathbb{R}$ while the distance between the output of machine i, $\Delta f_k^i(t)$ and its tracking reference $\Delta f^i_{ref_k}(t)$ is also minimum. To this end, we define the following performance index for single area non-homogeneous team optimal:

$$
\begin{aligned}
J_1^{NH}(L,G) := \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\Bigg[\sum_{k=1}^{3}\sum_{i_k=1}^{n_k}\Big[\alpha_k\Big(q_f\left(\Delta f_k^{i_k}(t)-\Delta f^{i_k}_{ref_k}(t)\right)^2+Ru_k^{i_k}(t)^2\Big)\Big] \\
+p_f\Big(\Delta\bar{f}(t)-\Delta\bar{f}_{ref}(t)\Big)^2+\tilde{p}\Big(\int_{\tau=0}^{t}(\Delta\bar{f}\tau)-\Delta\bar{f}_{ref}(\tau))d\tau\Big)^2 dt\Bigg],
\end{aligned}
\tag{3.3}
$$

where $\alpha_k = \frac{J_k}{\left[\sum_{j=1}^{3} n_j J_j\right]}$, $k=1,2,3$, and J_k denotes the kinetic energy of a typical machine in group k, while $\sum_{k=1}^{3}\alpha_k n_k = 1$. Finally $\Delta\bar{f}=\sum_{k=1}^{3}\sum_{i_k=1}^{n_k}\alpha_k\Delta f_k^{i_k}(t)$.

Problem 2. *The objective is to minimize the cost function* J_1^{NH} *under mean-field sharing information structure such that:*

$$
J_1^{NH^*} := J_1^{NH}(L^*,G^*) \leq J_1^{NH}(L,G). \tag{3.4}
$$

L,G *should be independent of the choice of reference trajectories.*

3.1.4 Comparison of Problem 1 and Problem 2

In this section, we shall be comparing the dynamic behaviour of single areas respectively under homogeneous and non-homogeneous conditions, as summarized in the table below.

Table 3.2 Comparison of A_1^H and A_1^{NH}

Case	A_1^H	A_1^{NH}
Controller	L^i, G	L_1^i, L_2^i, L_3^i, G
Observation	$y^i, \bar{y}$	$y_1^i, y_2^i, y_3^i, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{f}$
Model	Figure 3.4	Figure 3.6
Optimization	Problem 1	Problem 2

3.2 Modelling of Generators in Single Area Case

3.2.1 Areawise Generator Modelling

We shall start with the block diagram model in Figure 3.1 taken from [Tan, 2009]. All parameters/variables involved are defined in Table 3.1.
The plant for load frequency control consists of 4 parts:

— Governor with dynamics: $G_g = \frac{1}{T_G s+1}$
— Turbine generator with dynamics: $G_t = \frac{1}{T_T s+1}$
— Power systems with dynamics: $G_p = \frac{K_P}{T_P s+1}$
— Load disturbance P_d

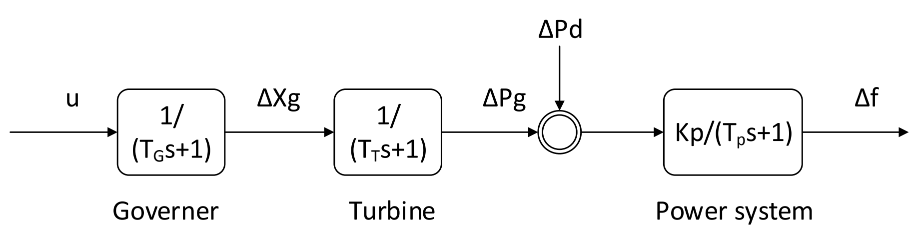

Figure 3.1 Linear model of generator

Proposition 1. *The 3 variables, which are correlated with each other in the following form:*

$$\begin{aligned} \Delta X_g(t) &= G_g u(t) \\ \Delta P_g(t) &= G_t \Delta X_g(t) \\ \Delta f(t) &= G_p(\Delta P_g(t) - \Delta P_d(t)) \end{aligned} \tag{3.5}$$

can be transformed to a state space representation:

$$\frac{d}{dt}\begin{bmatrix} \Delta f^{new}((t) \\ \Delta P^{new}(_g(t) \\ \Delta X^{new}(_g(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & -\frac{1}{T_G} \end{bmatrix} \begin{bmatrix} \Delta f^{new}((t) \\ \Delta P^{new}(_g(t) \\ \Delta X^{new}(_g(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_G} \end{bmatrix} u \tag{3.6}$$

where

$$\begin{aligned} \Delta f^{new}(t) &= \Delta f(t) \\ \Delta P_g^{new}(t) &= \Delta P_g(t) - \Delta P_d \\ \Delta X_g^{new}(t) &= \Delta X_g(t) \end{aligned} \tag{3.7}$$

Proof. By studying the relationship between variables marked in Table 3.1, the system dynamics of a generator can be represented as follows:

$$\frac{d}{dt}\begin{bmatrix} \Delta f(t) \\ \Delta P_g(t) \\ \Delta X_g(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & \frac{1}{T_G} \end{bmatrix} \begin{bmatrix} \Delta f(t) \\ \Delta P_g(t) \\ \Delta X_g(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_G} \end{bmatrix} u + \begin{bmatrix} -\frac{K_P}{T_P} \\ 0 \\ 0 \end{bmatrix} \Delta P_d \tag{3.8}$$

It is not a standard state space form, but in [Fosha and Elgerd, 1970],it is suggested that the effect of the (assumed constant) ΔP_d term be integrated in a redefinition of the state variables as deviations from their corresponding steady-state values, which, in our theory, is represented by the pre-set references.

$$\frac{d}{dt}\begin{bmatrix} \Delta f^{new}(t) \\ \Delta P_g^{new}(t) \\ \Delta X_g^{new}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & \frac{1}{T_G} \end{bmatrix} \begin{bmatrix} \Delta f^{new}(t) \\ \Delta P_g^{new}(t) \\ \Delta X_g^{new}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_G} \end{bmatrix} u \tag{3.9}$$

where

$$\begin{aligned} \Delta f^{new}(t) &= \Delta f(t) - \Delta f_{ref}(t) \\ \Delta P_g^{new} &= \Delta P_g(t) - \Delta P_{g_{ref}}(t) \\ \Delta X_g^{new} &= \Delta X_g(t) - \Delta X_{g_{ref}}(t) \end{aligned} \tag{3.10}$$

Furthermore, according to Assumption 1, we assume that ΔP_d is a very slowly changing variable whose rate of change is negligible compared with that of Δf. In steady state, $\Delta f = 0$, $\Delta P_g = 0$, $\Delta X_g = 0$, the system dynamics (3.6) will be as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & \frac{1}{T_G} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \Delta f_{ref}(t) \\ \Delta P_{gref}(t) - \Delta P_d \\ \Delta X_{gref}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_G} \end{bmatrix} 0 \tag{3.11}$$

Thus the reference for each variable could be computed:

$$\begin{aligned} \Delta f_{ref}(t) &= 0 \\ \Delta P_{gref}(t) &= \Delta P_d \\ \Delta X_{gref}(t) &= 0 \end{aligned} \tag{3.12}$$

□

3.2.2 Multi Homogeneous Machines Case

In reality, most power systems will be fed by multiple interconnected generators, where each machine has its own governor and turbine but the total load disturbance is a global signal. In this case the system could be represented by the left part of Figure 3.2, and taking advantage of linearity of the dynamics, we decompose the global disturbance and frequency deviation signals into individualized disturbance and frequency deviation signals to obtain the individualized representations of machines as in the right part of Figure 3.2.

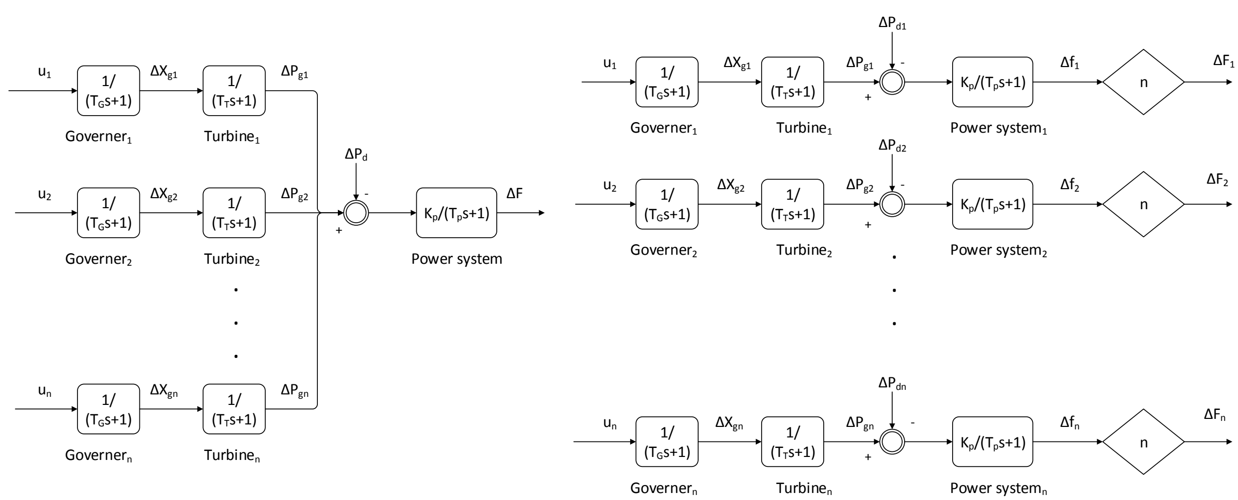

Figure 3.2 Generator modelling in single area multi homogeneous machines case

More specifically, as a preparation for the team formulation of the control problem, we separate the system into a sum of n individualized local Δf^i signals:

$$\Delta f(t) = \sum_{i=1}^{n} \Delta f^i(t) \tag{3.13}$$

with $\Delta f^i(t) = G_p(\Delta P_g^i(t) - \Delta P_d^i(t))$

Proof. Taking advantage of the homogeneous character, the load change is separated equally to each agents, so that $\Delta P_d^i(t) = \frac{1}{n}\Delta P_d(t)$
Besides, ΔP_g^i the incremental change in generator output for each generator $i \in \{1, \dots, n\}$ such that the total summation of incremental change in generator output is $\Delta P_g = \sum_{i=1}^{n} \Delta P_g^i$. Then:

$$\begin{aligned}
\Delta f(t) &= G_p(\sum_{i=1}^{n} \Delta P_g^i(t) - \Delta P_d(t)) \\
&= G_p \sum_{i=1}^{n} \left(\Delta P_g^i(t) - \frac{1}{n}\Delta P_d(t) \right) \\
&= G_p \sum_{i=1}^{n} \left(\Delta P_g^i(t) - \Delta P_d^i(t) \right) = \sum_{i=1}^{n} \Delta f^i(t)
\end{aligned} \tag{3.14}$$

□

Besides, as the separated subgroup is only a part of the total frequency, its magnitude will be much smaller than the original one, as the per unit base is taken from the areawise model. Then we consider it as the contribution of its ΔP_g^i to the overall frequency output contributed by this turbine; thus we could add all these small frequency signals up, and the "virtual" output frequency $\Delta F^i = n\Delta f^i$, where n is the number of agents in the system, could be in the same per unit base as the original frequency. But in this case our controller has only feedback from ΔX_g^i, ΔP_g^i and Δf^i to keep all variables in same order of magnitude.

3.2.3 Multi Non-homogeneous Machines Case

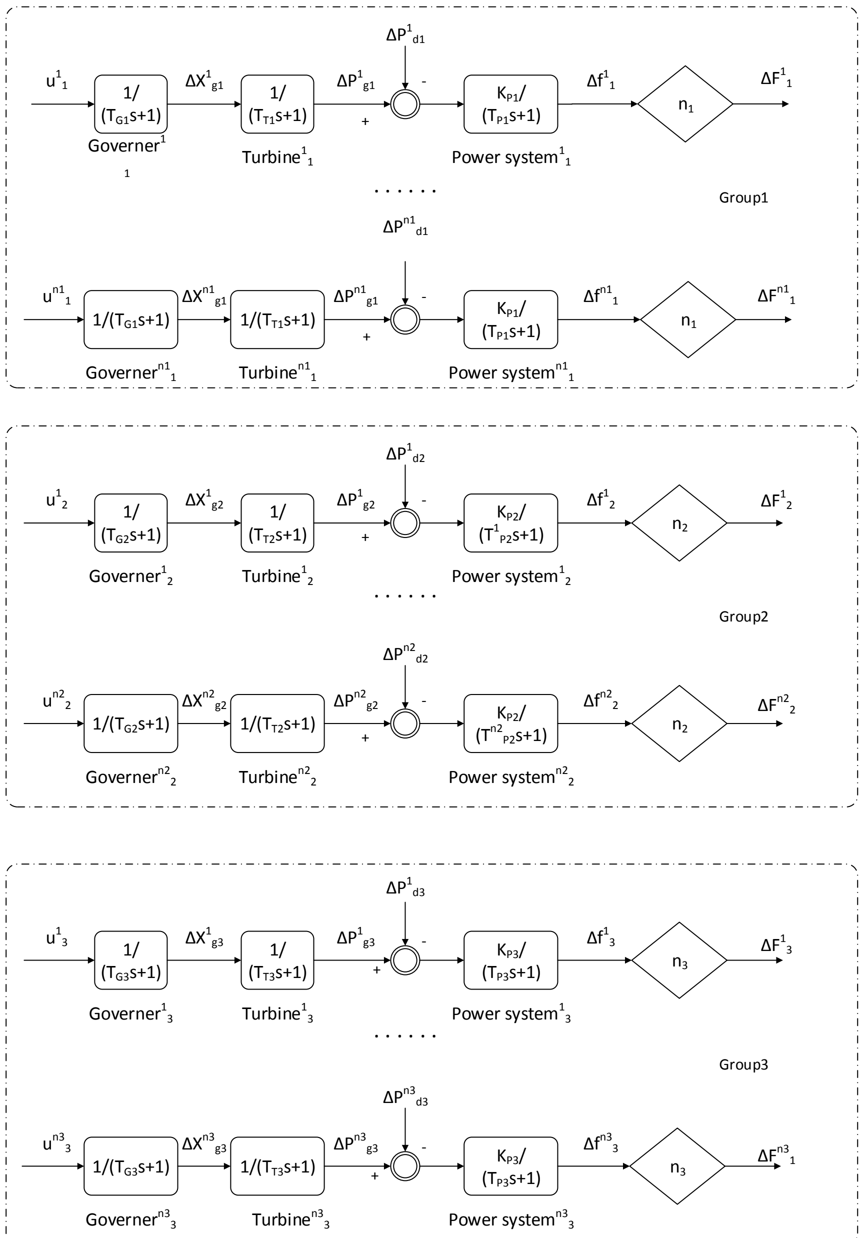

Figure 3.3 Generator modelling in single area multi non-homogeneous machines case

Here for the non-homogeneous case, we assume there exist 3 different groups $k \in \{1, 2, 3\}$, within which all agents are homogeneous, and representing small, medium and large generators, respectively. At first, proceeding as we did earlier in moving from left side to right side in Figure 3.2, we separate each of the subgroup homogeneous macromachine into n_k "virtual" multi homogeneous machines generation subsystems. Finally we combine all of them together to form a non-homogeneous team in Figure 3.3.

Besides, same as in Section 3.2.2, each small Δf_k^i will be multiplied by n_k if we want a 'virtual' frequency output in same per unit base with the original one while the controller has feed back only on Δf_k^i according to magnitude difference.

We present the modeling details in the following.

The three types of generators have different values of T_{G_k}, T_{Tk}, K_{Pk} while the values of T_{Pk} are inversely proportional to their angular moment of inertia, with $k \in \{1, 2, 3\}$ denote the machine class.

$$\begin{aligned}
\Delta f &= G_P(\sum_{k=1}^{3} \frac{J_k n_k}{\sum_{j=1}^{3} J_j n_j} \Delta \mathbf{P}_g) - G_P \Delta \mathbf{P}_d \\
&\Downarrow \\
\Delta f &= G_P(\sum_{k=1}^{3} \Delta \mathbf{P}_{g_k}) - G_P \Delta \mathbf{P}_d \\
&= G_P(\sum_{k=1}^{3} \Delta \mathbf{P}_{g_k} - \Delta \mathbf{P}_d) \\
&= G_P(\sum_{k=1}^{3} (\Delta \mathbf{P}_{g_k} - \Delta \mathbf{P}_{d_k}))
\end{aligned} \tag{3.15}$$

where

$$\Delta P_{dk} = \alpha_k n_k \Delta \mathbf{P}_d \tag{3.16}$$

$$\Delta P_{gk} = \alpha_k n_k \Delta \mathbf{P}_g \tag{3.17}$$

In the above, $\Delta \mathbf{P}_{g_k}$, $\Delta \mathbf{P}_{d_k}$, $k \in \{1, 2, 3\}$, are the aggregate generations of the three macromachines. Subsequently, the macromachine of the k^{th} homogeneous group, $k \in \{1, 2, 3\}$, is split into n_k individual machines. The final system is represented in Figure 3.3, where there exist 3 groups of n_k "virtual" generators. Then as in each group k agents are homogeneous:

$$\Delta P_{d_k}^i = \frac{1}{n_k} \Delta \mathbf{P}_{d_k} = \alpha_k \Delta \mathbf{P}_d \tag{3.18}$$

To simplify the presentation in Section 3.2.3 and Section 3.4.1, we shall assume that, in the case of a non-homogeneous single area, $i \in \{1, \ldots, n_1\}$ belongs to group 1, $i \in \{n_1 +$

$1, \ldots, n_1 + n_2\}$ belongs to group 2, and $i \in \{n_1 + n_2 + 1, \ldots, n\}$ belongs to group 3, where $n = n_1 + n_2 + n_3$.

3.3 Main Result for Problem 1

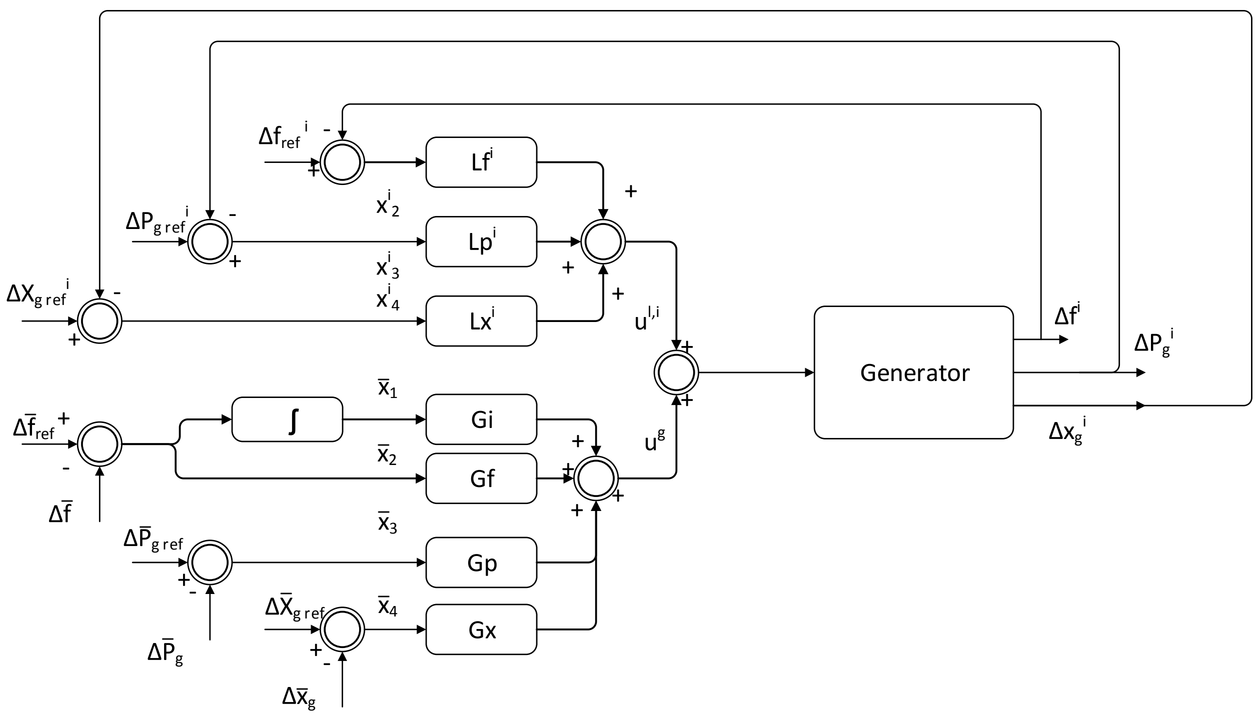

Figure 3.4 The control block diagram under MFS-IS for single area homogeneous system $i \in \{1, \ldots, n\}$, where the optimal gains are given by Theorem 1.

3.3.1 State-space Representation for A_1^H

Following [Das et al., 2013] and [Arabneydi, 2016], we construct a state-space representation for Problem 1.

Proposition 2. *Let Assumption 1 hold.*
There exists a state-space representation for the local controller as follows.

$$\begin{bmatrix} \dot{x}_1^i(t) \\ \dot{x}_2^i(t) \\ \dot{x}_3^i(t) \\ \dot{x}_4^i(t) \end{bmatrix} = A \begin{bmatrix} x_1^i(t) \\ x_2^i(t) \\ x_3^i(t) \\ x_4^i(t) \end{bmatrix} + Bu^i(t) + C + D \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} \tag{3.19}$$

where $x_1^i(t) = \int_{\tau=0}^{t} \Delta \bar{f}(\tau) d\tau$*, is the same for all individuals and* $x_j^i, j \in \{2,3,4\}$ *are defined as the difference between* Δf^i*,* ΔP_g^i*,* ΔX_g^i *and their references. Besides,* $\bar{x}_j, j \in \{1,2,3,4\}$ *is the average of* $x_j^i, j \in \{1,2,3,4\}$ *with*

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & 0 & \frac{1}{T_G} \end{bmatrix}, B := \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{T_G} \end{bmatrix}, D := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$C := -A \begin{bmatrix} 0 \\ 0 \\ \Delta P_d^i(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{K_P}{T_P} \Delta P_d^i(t) \\ \frac{1}{T_T} \Delta P_d^i(t) \\ 0 \end{bmatrix} \tag{3.20}$$

Proof. Note first that in keeping with the current PID strategy for the LFC in current power systems, we have augmented the individual states with a mean state, the integral of the frequency tracking error state. The first state equation is easily obtained from the definition of $\bar{x}_1$. Then It follows from (3.6) and Assumption 1 that

$$\begin{bmatrix} \dot{x}_2^i(t) \\ \dot{x}_3^i(t) \\ \dot{x}_4^i(t) \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \Delta f_{ref}^i(t) - \Delta f^i(t) \\ \Delta P_{gref}^i(t) - \Delta P_g^i(t) \\ \Delta X_{gref}^i(t) - \Delta X_g^i(t) \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} -\Delta f^i(t) \\ -\Delta P_g^i(t) \\ -\Delta X_g^i(t) \end{bmatrix} \tag{3.21}$$

where the derivatives of the reference trajectory are zero because they are assumed to be constant according to Assumption 1.
According to (3.6),

$$\begin{bmatrix} \dot{x}_2^i(t) \\ \dot{x}_3^i(t) \\ \dot{x}_4^i(t) \end{bmatrix} = - \begin{bmatrix} -\frac{1}{T_P} & \frac{K_P}{T_P} & 0 \\ 0 & -\frac{1}{T_T} & \frac{1}{T_T} \\ 0 & 0 & \frac{1}{T_G} \end{bmatrix} \begin{bmatrix} \Delta f(t) \\ \Delta P_g(t) \\ \Delta X_g(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_G} \end{bmatrix} u \tag{3.22}$$

The relationship between $x_j, j \in \{2,3,4\}$ is found by a simple change of variable

And the reference value is the same as in (3.7).

□

Proposition 3. *Let Assumption 1 hold. There exists a state-space representation for the* *global controller(i.e. a controller acting only on mean field quantities) as follows.*

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \\ \dot{\bar{x}}_3(t) \\ \dot{\bar{x}}_4(t) \end{bmatrix} = (A+D) \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} + B\bar{u}(t) + \bar{C} \tag{3.23}$$

where $\bar{x}_j = \frac{1}{n}\sum_{i=1}^n x_j^i$ *and* $\bar{u} = \frac{1}{n}\sum_{i=1}^n u^i$.

Proof. By averaging of (3.6) over all n systems, the result is obtained modulo an obvious reindexing of the state variables.

□

Given the quadratic cost function in (3.1), we shall establish that the optimal control action of system i at time t can be expressed in terms of state-space representations in Propositions 2 and 3 and block diagram in Figure 3.4, more specifically as:

$$u^i(t) \;=\; L_f^i x_2^i(t) + L_p^i x_3^i(t) + L_x^i x_4^i(t) + G_I\bar{x}_1(t) + G_f\bar{x}_2(t) + G_p\bar{x}_3(t) + G_x\bar{x}_4(t). \tag{3.24}$$

Proof. There is no feedback gain on x_1^i because it is equal to $\bar{x}_1$ so that the gain is only applied on mean field level. □

In building up to the above result, we use the following proposition proved in [Arabneydi, 2016]. It establishes that the performance index (3.1) can be expressed in terms of the state-space variables in Propositions 2 and 3.

Proposition 4. *Suppose Assumption 1 holds. For any* (L, G), *the previous cost function* (3.1) *can be written in a quadratic form as below with a specified weighting matrix* Q *and* $\tilde{Q}$

$$\begin{aligned} J_1^H(L,G) = \lim_{T\to\infty}\Bigg[\frac{1}{T}\int_{t=0}^{T}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\Bigg(&\begin{bmatrix} x_1^i(t) \\ x_2^i(t) \\ x_3^i(t) \\ x_4^i(t) \end{bmatrix}^\intercal Q \begin{bmatrix} x_1^i(t) \\ x_2^i(t) \\ x_3^i(t) \\ x_4^i(t) \end{bmatrix} + Ru^i(t)^2\Bigg) \\ &+ \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix}^\intercal \tilde{Q} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix}\Bigg) dt\Bigg], \end{aligned} \tag{3.25}$$

where

$$Q := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{Q} := \begin{bmatrix} \tilde{p} & 0 & 0 & 0 \\ 0 & p_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

.

We further impose the following assumption on the dynamics and cost.

Assumption 2. *(A, B) and $(\tilde{A}, \tilde{B})$ are stabilizable and $(A, Q^{1/2})$ and $(\tilde{A}, \tilde{Q}^{1/2})$ are detectable.*

The rest of the developments in this chapter are based on the methods of [Arabneydi, 2016] in Appendix A.

3.3.2 Solution for Problem 1

At first, we define an auxiliary model that is obtained based on the state variables introduced in Section 3.3.1 with the same cost function:

Define $\bar{x}_j(t) := \frac{1}{n}\sum_{i=1}^{n} x_j^i(t)$ and $\breve{x}_j^i(t) := x_j^i(t) - \bar{x}_j(t)$ for $j \in \{1,2,3,4\}$. In addition, let $\Delta\bar{f}_{ref}(t) := \frac{1}{n}\sum_{i=1}^{n} \Delta f_{ref}^i(t)$. Furthermore, define $\breve{u}^i(t) := u^i(t) - \bar{u}(t)$. In this case the previous system becomes a *single centralized agent* that controls the average $\bar{u}$ and N agents that control $\breve{u}^i$'s, which are differences between each u^i and the mean field $\bar{u}$. The system dynamic of mean field is same as in Proposition 3 and that of $\breve{x}$ is as follows:

Proposition 5. *For any $i \in \{1, \dots, n\}$,*

$$\begin{bmatrix} \dot{\breve{x}}_1^i(t) \\ \dot{\breve{x}}_2^i(t) \\ \dot{\breve{x}}_3^i(t) \\ \dot{\breve{x}}_4^i(t) \end{bmatrix} = A \begin{bmatrix} \breve{x}_1^i(t) \\ \breve{x}_2^i(t) \\ \breve{x}_3^i(t) \\ \breve{x}_4^i(t) \end{bmatrix} + B\breve{u}^i(t) \tag{3.26}$$

Proof. The auxiliary model is of following form by subtraction of (3.19) and (3.23):

$$\begin{bmatrix} \dot{\breve{x}}_1^i(t) \\ \dot{\breve{x}}_2^i(t) \\ \dot{\breve{x}}_3^i(t) \\ \dot{\breve{x}}_4^i(t) \end{bmatrix} = A \begin{bmatrix} \breve{x}_1^i(t) \\ \breve{x}_2^i(t) \\ \breve{x}_3^i(t) \\ \breve{x}_4^i(t) \end{bmatrix} + B\breve{u}^i(t) + C - \bar{C} \tag{3.27}$$

where C defined as the constant in Proposition 2
Thus, $C - \bar{C} = 0$ and (3.26) is obtained. $\square$

Lemma 1. *For all i ,*

$$\frac{1}{T}\int_{t=0}^{T}\frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} x_1^i(t) \\ x_2^i(t) \\ x_3^i(t) \\ x_4^i(t) \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} x_1^i(t) \\ x_2^i(t) \\ x_3^i(t) \\ x_4^i(t) \end{bmatrix} + Ru^i(t)^2\right)dt$$

$$= \frac{1}{T}\int_{t=0}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} \breve{x}_1^i(t) \\ \breve{x}_2^i(t) \\ \breve{x}_3^i(t) \\ \breve{x}_4^i(t) \end{bmatrix}^{T} Q \begin{bmatrix} \breve{x}_1^i(t) \\ \breve{x}_2^i(t) \\ \breve{x}_3^i(t) \\ \breve{x}_4^i(t) \end{bmatrix} + (\breve{u}^i)^T R\breve{u}^i\right)\right)dt \qquad (3.28)$$

$$+ \frac{1}{T}\int_{t=0}^{T}\left(\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix}^{T} Q \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} + \bar{u}^T R\bar{u}\right)dt$$

Proof. Lemma 1 is proved by simple calculation of $\sum_i \breve{X}^i = 0$ $\square$

Based on Proposition 3 and Lemma 1, we are interested in feedback strategies (L, G), that minimize the following performance index:

$$
\begin{aligned}
J_1^H(L,G) &:= \lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\begin{bmatrix}\breve{x}_1^i(t)\\ \breve{x}_2^i(t)\\ \breve{x}_3^i(t)\\ \breve{x}_4^i(t)\end{bmatrix}^T Q \begin{bmatrix}\breve{x}_1^i(t)\\ \breve{x}_2^i(t)\\ \breve{x}_3^i(t)\\ \breve{x}_4^i(t)\end{bmatrix} + (\breve{u}^i)^T R\breve{u}^i\right]\right)dt\right.\\
&\left.+\frac{1}{T}\int_{t=0}^{T}\left(\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix}^T Q \begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix} + \bar{u}^T R\bar{u} + \begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix}^T \tilde{Q} \begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix}\right)dt\right]\\
&= \lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\begin{bmatrix}\breve{x}_1^i(t)\\ \breve{x}_2^i(t)\\ \breve{x}_3^i(t)\\ \breve{x}_4^i(t)\end{bmatrix}^T Q \begin{bmatrix}\breve{x}_1^i(t)\\ \breve{x}_2^i(t)\\ \breve{x}_3^i(t)\\ \breve{x}_4^i(t)\end{bmatrix} + (\breve{u}^i)^T R\breve{u}^i\right]\right)dt\right.\\
&\left.+\frac{1}{T}\int_{t=0}^{T}\left(\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix}^T (Q+\tilde{Q}) \begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \bar{x}_3(t)\\ \bar{x}_4(t)\end{bmatrix} + \bar{u}^T R\bar{u}\right)dt\right]
\end{aligned}
\tag{3.29}
$$

Theorem 1. *Under Assumptions 1 , 2, the optimal strategy (L^*, G^*) of Problem 1 in A_1^H is given as follows:*

$$
\breve{u}^i(t) = \breve{K}\begin{bmatrix}\breve{x}_1^i(t)\\ \breve{x}_2^i(t)\\ \breve{x}_3^i(t)\\ \breve{x}_4^i(t)\end{bmatrix}, \quad \bar{u}(t) = \bar{K}\begin{bmatrix}\bar{x}_1^i(t)\\ \bar{x}_2^i(t)\\ \bar{x}_3^i(t)\\ \bar{x}_4^i(t)\end{bmatrix} \tag{3.30}
$$

optimal gains are given by

$$
\begin{aligned}
\breve{K} &= \text{Feedback}(A, B, Q, R),\\
\bar{K} &= \text{Feedback}((A+D), B, (Q+\tilde{Q}), R).
\end{aligned}
\tag{3.31}
$$

Thus, optimal gains are the following:

$$
\begin{aligned}
&{L_f^i}^* = \breve{K}_2, \quad {L_p^i}^* = \breve{K}_3, \quad {L_x^i}^* = \breve{K}_4,\\
&{G_I}^* = \bar{K}_1, \quad {G_f}^* = \bar{K}_2 - \breve{K}_2, \quad {G_p}^* = \bar{K}_3 - \breve{K}_3, \quad {G_x}^* = \bar{K}_4 - \breve{K}_4
\end{aligned}
\tag{3.32}
$$

Proof. From standard results in optimal control, the solution of the above decoupled LQ problems are given by standard algebraic Riccati equations, which give (3.30).

Besides, as x_1^i is the same for all individuals, $\breve{x}_1^i$ is clearly 0 among all agents, which results in effective elimination of the $\breve{K}_1$ term in the controller.

Now, from the definition of $\breve{u}^i(t)$, $u^i(t) = \breve{u}^i(t) + \bar{u}(t)$. Thus the optimal strategy of Problem 1 provided by (3.30) can be rewritten as follows:

$$\begin{aligned} u^i(t) &= \breve{K}_1\breve{x}_1^i(t) + \breve{K}_2\breve{x}_2^i(t) + \breve{K}_3 x_3^i(t) + \breve{K}_4 x_4^i(t) \\ &\quad + \bar{K}_1\bar{x}_1(t) + \bar{K}_2\bar{x}_2(t) + \bar{K}_3\bar{x}_3(t) + \bar{K}_4\bar{x}_4(t) \\ &= \breve{K}_2 x_2^i(t) + \breve{K}_3 x_3^i(t) + \breve{K}_4 x_4^i(t) \\ &\quad + \bar{K}_1\bar{x}_1(t) + (\bar{K}_2 - \breve{K}_1)\bar{x}_2(t) + (\bar{K}_3 - \breve{K}_2)\bar{x}_3(t) + (\bar{K}_4 - \breve{K}_3)\bar{x}_4(t) \end{aligned} \tag{3.33}$$

Here (3.33) is in a form equivalent to (3.24). According to dynamics (3.26) and (3.23), and the definition of cost function $J_1^H(g)$, we have $n+1$ decoupled LQ problems where n of them have dynamics in the form of (3.26) and quadratic cost with Q and R weighting matrices. The remaining LQ problem (i.e., the $(n+1)^{th}$ one) has dynamics (3.23).

Since the auxiliary centralized system has the same cost function as the original decentralized system, the optimal control law for the auxiliary system should be also optimal for the original one. Therefore,it should also be optimal for Problem 1. □

3.3.3 Numerical Results for Problem 1

In this section, we present some numerical results for ideal case, where these typical values of the system parameters are taken from [Tan, 2009, 2011].

Consider a homogeneous population of n generators, where the distribution of initial difference of frequency Δf is uniform between -0.04 and 0.08 Hz(which is close to the value of original output frequency divided by n ,if represented in p.u.Hz), and that of ΔP_g and ΔX_g are also uniformly distributed around 0 between -0.02/n and 0.02/n p.u.MW(because the per unit base used here are the areawise base so that it is separated equally for each individual). We wish to achieve the objective of driving all these variables to 0 thus having these variables attain their nominal values. Other parameters used in the simulation are as follows:

$$p_f = q_f = 20, \quad \tilde{p} = 30, \quad R_1 = R_2 = 15 \tag{3.34}$$

Parameters of each homogeneous generator, taken from [Tan, 2009], are the following:

$$K_P = 120 \quad T_P = 20 \quad T_T = 0.3 \quad T_G = 0.08 \tag{3.35}$$

So that:

$$G_g = \frac{1}{0.08s + 1} \quad G_t = \frac{1}{0.3s + 1} \quad G_p = \frac{120}{20s + 1} \tag{3.36}$$

100 systems starting from initial random values are taken into simulation and with the objective of having their mean frequency deviations from nominal converge to zero $\Delta\bar{f}_{ref} = 0$. As shown in Figure 3.5, 20 samples of the trajectories are represented. Besides, the tracking error of total frequency output is also shown for clear verification. Positive load disturbances, i.e. load increases, are added at different instants, namely $\frac{T}{4}$, $\frac{T}{3}$, which are marked by vertical lines.

In the following graph, we plot the small output frequency of all 'virtual' systems in Hz and the total output in p.u.Hz with a base of 60 Hz to avoid too small numbers and thus make the graph clearer.

In the single area homogeneous case, we compare this team optimal control theory with normal LQR control method, sharing the same cost function with the team optimal control theory that we used, which take into concern only areawise generator model with objective of maintaining only average frequency to 0, so that every agent has the same controller.

Supposing that from the beginning, an load change hit the generator network, thus resulting in individual machines having different initial conditions, with an initial mean frequency at 0.03 Hz (total tracking error 0.1 p.u.Hz). Then at time $\frac{T}{4}$ and $\frac{T}{3}$, 2 short impulsive disturbances around 0.02/n p.u.MW, which also hits the generator system thus creates a different load change for different machines, has been added to the system, with the assumption that load changes are slowly changing variable.

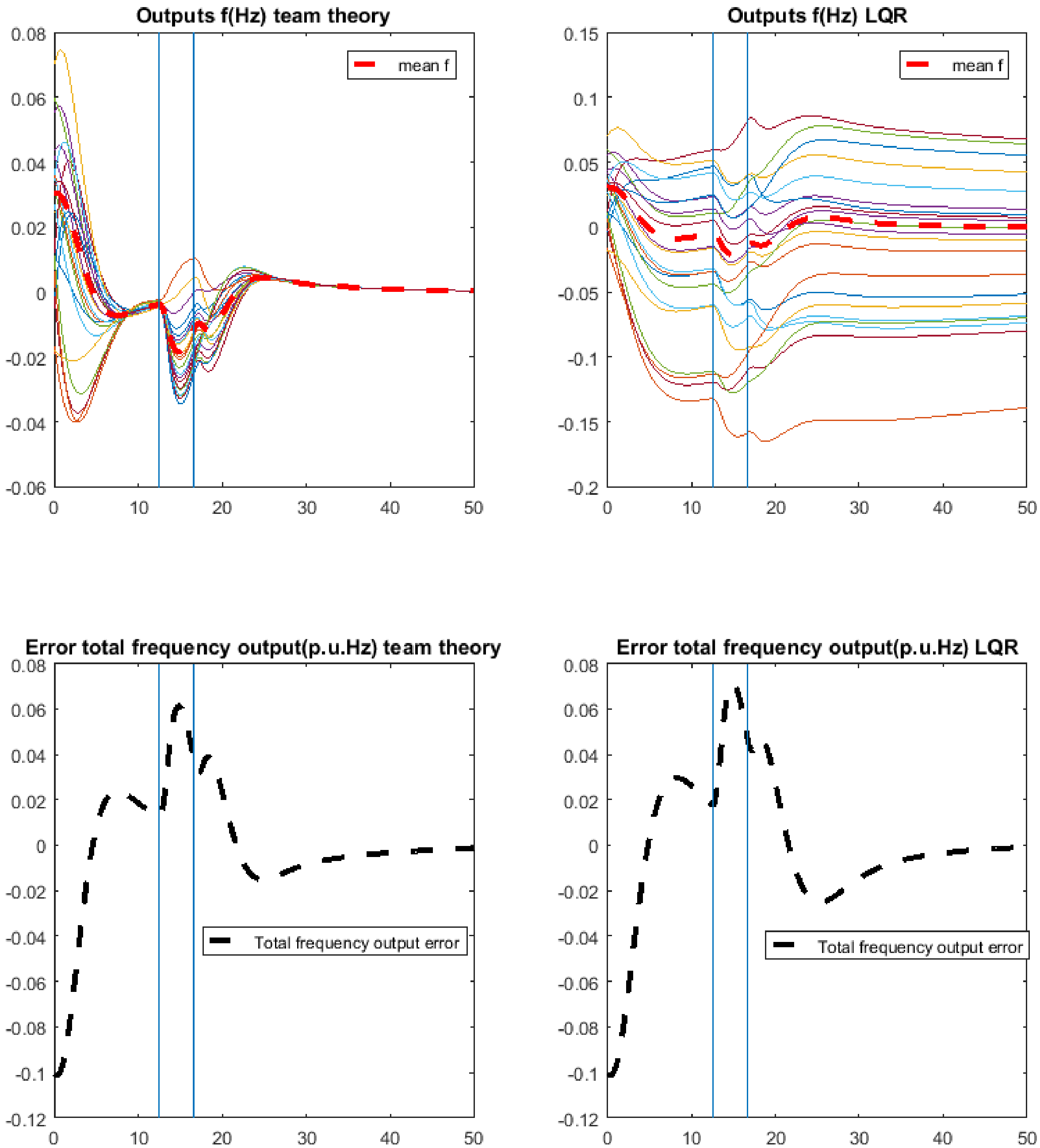

Figure 3.5 Comparison of 20 randomly selected homogeneous agents of single area control systems under team optimal control theory (left) and normal LQR method (right).

In Figure 3.5, we have 4 graphs where the two on the left side represent individual macromachine performances(top) and total frequency error(below) using team optimal control theory, while the two on the right side represent those of the normal LQR method, however with the controls computed based on macromachine representations.

Through comparison of the 2 graphs below, we could observe the convergence to 0 of (mean) frequency tracking error in both cases, which shows the effectiveness of both methods in areawise load frequency control.

However, when focusing on the 2 upper graphs, a significant difference emerges in the individualized machine behaviours. In this case the normal LQR method loses convergence, meaning that the initial differences between individual machine states are not eliminated; thus these machines are not operating at their nominal value. Furthermore, the differences between machine frequencies are even increased when facing disturbance.

However, team theory has done a much better job in this respect. With an additional local controller, this control approach is driving each individual agent towards the desired mean-field frequency, thus enabling all agents to work at nominal value. No matter when a disturbance appears, causing each individual machine to deviate from nominal value, the frequency error could be adjusted quickly, thus ensuring a more desirable working environment for generators, and verifying the usefulness of individualized local controllers.

In conclusion, by applying the controller based team theory, we have given each individual machine the possibility to remain closer to its nominal value. Thus, all individual frequency deviations could remain within 0.15 p.u for the areawise system, the limit beyond which synchronous machines may loose synchronism.

3.4 Main Results for Problem 2

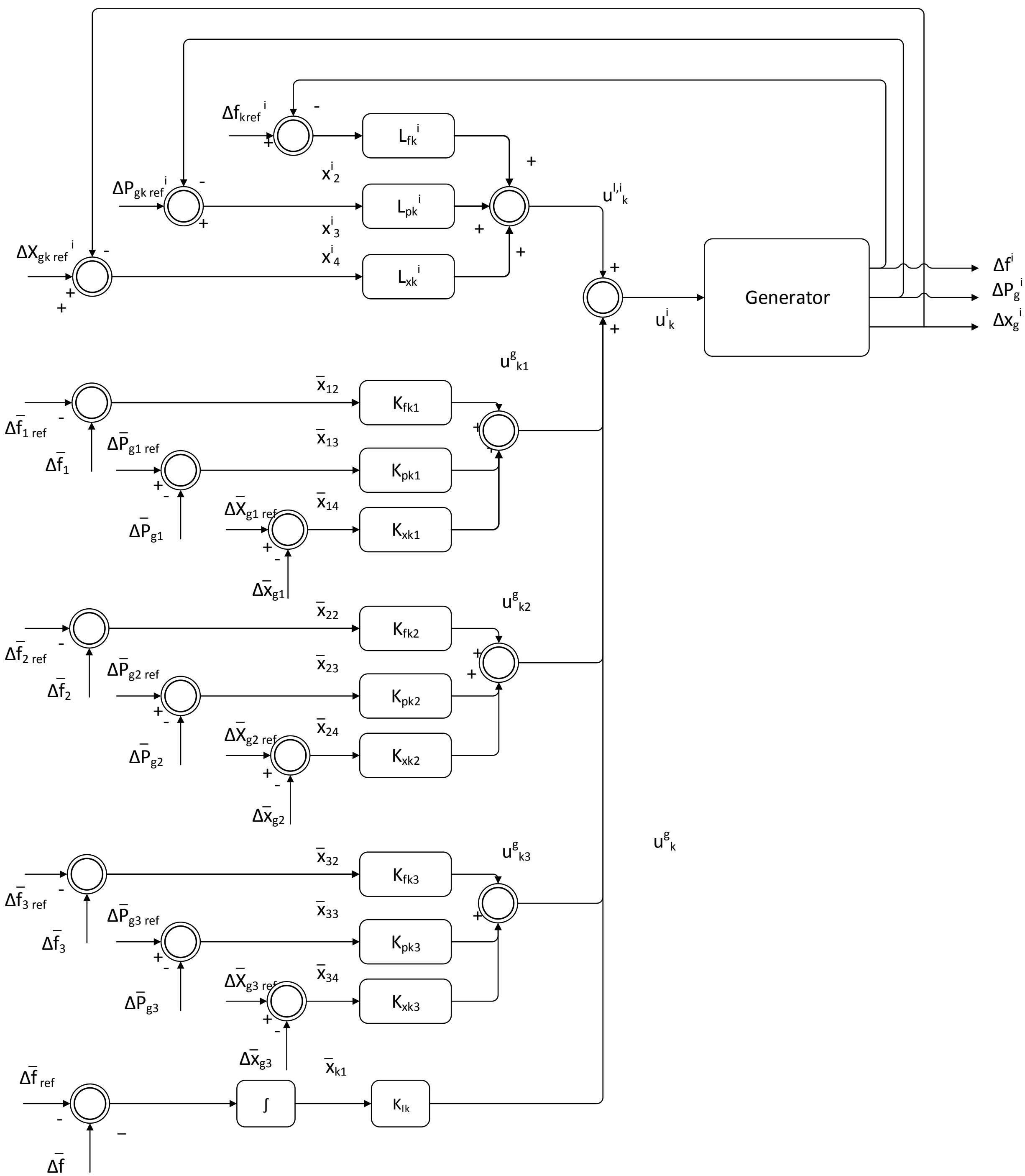

Figure 3.6 The control block diagram under MFS-IS for single area non homogeneous system $i \in \{1, \ldots, n\}$, where the optimal gains are given by Theorem 2.

3.4.1 State-space Representation for A_1^{NH}

Table 3.3 Notation of variables used for single area non homogeneous LFC

Notation used for agent $i_k \in n_k$ in group $k \in \{1, 2, 3\}$

Symbol	Description
$x_1^{i_k}$	$\int_{\tau=0}^{t}(\Delta\bar{f}_{ref}(\tau)) - \Delta\bar{f}(\tau))d\tau$
$x_2^{i_k}$	$\Delta f_{ref_k}^{i_k} - \Delta f_k^{i_k}$
$x_3^{i_k}$	$\Delta P_{gref_k}^{i_k} - \Delta P_{g_k}^{i_k}$
$x_4^{i_k}$	$\Delta X_{gref_k}^{i_k} - \Delta X_{g_k}^{i_k}$
$X_k^{i_k}$	$[x_1^{i_k}(t)\ x_2^{i_k}(t)\ x_3^{i_k}(t)\ x_4^{i_k}(t)]^T$

Notation used for mean-field of group $k \in \{1, 2, 3\}$

Symbol	Description
$\bar{x}_1$	$\sum_{i_k=1}^{n_k} x_1^{i_k} = \int_{\tau=0}^{t}(\Delta\bar{f}_{ref}(\tau)) - \Delta arf(\tau))d\tau$
$\bar{x}_2^k$	$\sum_{i_k=1}^{n_k} x_2^{i_k} = \Delta\bar{f}_{ref_k} - \Delta\bar{f}_k$
$\bar{x}_3^k$	$\sum_{i_k=1}^{n_k} x_3^{i_k} = \Delta\bar{P}_{gref_k} - \Delta\bar{P}_{g_k}$
$\bar{x}_4^k$	$\sum_{i_k=1}^{n_k} x_4^{i_k} = \Delta\bar{X}_{gref_k} - \Delta\bar{X}_{g_k}$
$\bar{X}_k$	$\sum_{i_k=1}^{n_k} X_k^{i_k} = [\bar{x}_1(t)\ \bar{x}_2^k(t)\ \bar{x}_3^k(t)\ \bar{x}_4^k(t)]^T$

Notation used for mean-field of the entire population

Symbol	Description
$\bar{X}$	$\sum_{k=1}^{3}\sum_{i_k=1}^{n_k} \alpha_k X_k^{i_k} = \sum_{k=1}^{3} \alpha_k n_k \bar{X}_k$
$\tilde{X}$	$[\bar{X}_1\ \bar{X}_2\ \bar{X}_3]^T$

Following [Das et al., 2013] and [Arabneydi, 2016], we construct a state-space representation for Problem 2 for A_1^{NH}.

Proposition 6. *Let Assumption 1 hold. Also, assume there exist 3 groups which contain n_1, n_2,n_3 agents respectively representing small, medium and large generators, define $X_k^i := [x_1^{i_k}(t)\ x_2^{i_k}(t)\ x_3^{i_k}(t)\ x_4^{i_k}(t)]^T$ for agent i_k in subgroup $k \in \{1, 2, 3\}$, where $x_1^{i_k} = \int_{\tau=0}^{t}(\Delta\bar{f}_{ref}(\tau)) - \Delta\bar{f}(\tau))d\tau$ which is the integral of the total average of all individual frequency states, and is the same for all agents in all groups, while for $i_k \in$ group k, $x_j^{i_k}, j \in \{2, 3, 4\}$ denotes the differences between $\Delta f_k^{i_k}$, $\Delta P_{g_k}^{i_k}$, $\Delta X_{g_k}^{i_k}$ and their references.*

There exists a state-space representation for a generator $i_k \in$ group k:

$$\begin{aligned} X_k^{i_k} &:= \begin{bmatrix} \dot{x}_1^{i_k}(t) \\ \dot{x}_2^{i_k}(t) \\ \dot{x}_3^{i_k}(t) \\ \dot{x}_4^{i_k}(t) \end{bmatrix} \\ &= A_k \begin{bmatrix} x_1^{i_k}(t) \\ x_2^{i_k}(t) \\ x_3^{i_k}(t) \\ x_4^{i_k}(t) \end{bmatrix} + B_k u_k^{i_k}(t) + C_k + D \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \\ \bar{X}_3(t) \end{bmatrix} \\ &= A_k X_k^{i_k} + B_k u_k^{i_k}(t) + C_k + D\tilde{X} \end{aligned} \tag{3.37}$$

with $\tilde{X} = \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \\ \bar{X}_3(t) \end{bmatrix}$ *and* $\bar{X}_k = \frac{1}{n_k}\sum_{i_k=1}^{n_k} X_k^{i_k}$

with the above state space matrices as follows:

$$\begin{aligned} A_k &:= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \\ 0 & 0 & -\frac{1}{T_{T_k}} & \frac{1}{T_{T_k}} \\ 0 & 0 & 0 & \frac{1}{T_{G_k}} \end{bmatrix}, B_k := \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{T_{G_k}} \end{bmatrix}, \\ C_k &:= -A_k \begin{bmatrix} 0 \\ 0 \\ \Delta P^i_{g_k ref}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{K_P}{T_P}\Delta P^i_{d_k}(t) \\ \frac{1}{T_T}\Delta P^i_{d_k}(t) \\ 0 \end{bmatrix}, \\ D_k &:= \begin{bmatrix} 0 & \alpha_1 n_1 & 0 & 0 & 0 & \alpha_2 n_2 & 0 & 0 & 0 & \alpha_3 n_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{3.38}$$

Proof. Considering that the mean area frequency $\bar{\omega}$ is related to the total kinetic energy of all rotating machines, one can write:

$$\frac{1}{2}(\sum_{k=1}^{3} n_k J_k)\bar{\omega}^2 = \sum_{k=1}^{3}\sum_{i_k=1}^{n_k}\frac{1}{2}J_k\omega_{i_k}^2 \tag{3.39}$$

where J_k is the angular moment of inertia of a machine in group k. Thus, the values ω_{ik}

could be considered identical in group k, and all group frequencies are close to a nominal ω_0, one can after linearisation rewrite approximately the above equation to obtain:

$$(\sum_{k=1}^{3} n_k J_k)(\omega_0^2 + 2\omega_0 \Delta\bar{\omega}) = \sum_{k=1}^{3} J_k n_k (\omega_0^2 + 2\omega_0 \Delta\omega_k) \tag{3.40}$$

which finally leads to the system mean frequency equation:

$$\Delta\bar{\omega} = \sum_{k=1}^{3} \sum_{i_k=1}^{n_k} \alpha_k \Delta\omega_{i_k} = \sum_{k=1}^{3} \alpha_k n_k \Delta\bar{\omega}_k \tag{3.41}$$

and correspondingly, by integrating frequency over time one obtains the first state component as:

$$\bar{x}_1 = \sum_{k=1}^{3} \alpha_k n_k \bar{x}_1^k(t) \tag{3.42}$$

$$\bar{x}_2 = \sum_{k=1}^{3} \alpha_k n_k \bar{x}_2^k(t) \tag{3.43}$$

with $\bar{x}_2$ and $\bar{x}_2^k$ respectively denoting the weighted overall average first state component, and group k average first state component, while $\bar{x}_1$ and $\bar{x}_1^k$ denote their integral, receptively.

Thus the dynamics of x_1^i in group k could be established via differentiation.

For the rest of the states, the derivations follow from Proposition 2. □

Proposition 7. *Let Assumption 1 hold.*
The dynamics of the overall system could be expressed as follows:

$$\dot{\tilde{X}} := \left(\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} + \begin{bmatrix} D \\ D \\ D \end{bmatrix} \right) \tilde{X} + \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} + \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \end{bmatrix} \tag{3.44}$$

Proof. By averaging of (3.37) over all n_k systems,we get the following equation:

$$\dot{\bar{X}}_k := A_k \bar{X}_k + B_k \bar{u}_k(t) + \bar{C}_k + D\tilde{X} \tag{3.45}$$

where $\bar{X}_k = \frac{1}{n_k} \sum_{i_k=1}^{n_k} X_k^{i_k}$

Then as $\tilde{X} = \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \\ \bar{X}_3(t) \end{bmatrix}$, the system dynamic of $\tilde{X}$ is obtained by simple combination of system dynamic of $\bar{X}_k$

$\square$

The control action of system i at time t can be expressed in terms of state-space representations in Propositions 6 and 7 and block diagram in Figure 3.6, i.e.,

$$u_k^i(t) = L_k^i X_k^i + G_k \tilde{X} \tag{3.46}$$

Proof. This non homogeneous system is composed of 3 homogeneous groups, representing large, medium size and small generators, respectively. In each group all agents are homogeneous, where a control action similar to (3.24) can be applied as $\bar{u}_k = \sum_{i_k=1}^{n_k} u_k^{i_k}$. But in order to avoid introducing integral control on the frequency deviation states of every homogeneous subgroup (which would eventually lead to saturation), we rewrite the dynamics so that only the global weighted frequency deviation is introduced as a common extra state in every individual state equation. This eventually leads to a feedback which involves integral control on the overall mean frequency deviation and thus state feedback only on $x_1^{i_k}$ which is common to all machines. $\square$

In the next Proposition, it is shown that the performance index (3.1) can be expressed with respect to the state-space formulations of Propositions 6 and 7.

Proposition 8. *Suppose Assumption 1 holds. For any (L, G), $J_1^{NH}(L, G)$ could be rewritten into following quadratic form:*

$$J_1^{NH}(L,G) =$$
$$\lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(\sum_{k=1}^{3}\sum_{i_k=1}^{n_k}\alpha_k\left(\begin{bmatrix} x_1^{i_k}(t) \\ x_2^{i_k}(t \\ x_3^{i_k}(t) \\ x_4^{i_k}(t)\end{bmatrix}^{\intercal} Q \begin{bmatrix} x_1^{i_k}(t) \\ x_2^{i_k}(t) \\ x_3^{i_k}(t) \\ x_4^{i_k}(t)\end{bmatrix} + R_k u^{i_k}(t)^2\right) + \bar{X}^{\intercal}\tilde{Q}\bar{X}\right) dt\right], \tag{3.47}$$

where

$$Q := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{Q} := \begin{bmatrix} \tilde{p} & 0 & 0 & 0 \\ 0 & p_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof. (3.47) is transformed from (3.3) by noting that $\bar{X} = \sum_{k=1}^{3} \alpha_k n_k \bar{X}_k$. $\square$

In the following, we impose the following assumption on the dynamics and cost.

Assumption 3. *(A_k, B_k) and $(\tilde{A}_k, \tilde{B}_k)$ are stabilizable and $(A_k, Q^{1/2})$ and $(\tilde{A}_k, \tilde{Q}^{1/2})$ are detectable.*

Assumption 4. *For every t, Q_k, $\tilde{Q}$ and R_k are symmetric matrices that satisfy*

$$\begin{aligned} Q_k \geq 0, \forall k, \quad & \begin{bmatrix} \alpha_1 n_1 Q_1 & 0 & 0 \\ 0 & \alpha_2 n_2 Q_2 & 0 \\ 0 & 0 & \alpha_3 n_3 Q_3 \end{bmatrix} \geq 0 \\ R_k > 0, \forall k, \quad & \begin{bmatrix} \alpha_1 n_1 R_1 & 0 & 0 \\ 0 & \alpha_2 n_2 R_2 & 0 \\ 0 & 0 & \alpha_3 n_3 R_3 \end{bmatrix} > 0 \end{aligned} \tag{3.48}$$

Here we chose Q_k and R_k to be the same for all $k \in \{1, 2, 3\}$.

3.4.2 Solution for Problem 2

At first, we define an auxiliary model that is isomorphic to the model presented in Section 3.4.1 with the same cost function:
For each homogeneous subgroup, define $\bar{x}_j^k(t) := \frac{1}{n_k} \sum_{i_k=1}^{n_k} x_j^{i_k}(t)$ and $\breve{x}_j^{i_k}(t) := x_j^{i_k}(t) - \bar{x}_j^k(t)$ for $j \in \{1, 2, 3, 4\}$. In addition, let $\Delta \bar{f}_{ref}(t) := \sum_{k=1}^{3} \sum_{i_k=1}^{n_k} \alpha_k \Delta \bar{f}_{ref_k}^{i_k}(t)$, and define $\breve{u}_k^{i_k}(t) := u_k^{i_k}(t) - \bar{u}_k(t)$. Thus, we define $\breve{X}_k = X_k - \bar{X}_k$.

In this case the previous system becomes the combination of a *single centralized agent* that controls the average input $\bar{u}_k$, together with n_k agents that control $\breve{u}_k$, as defined above. The system dynamics of the mean field is the same as in Proposition 7 and that of $\breve{X}_k$ is detailed in Proposition 9 below.

Proposition 9. *According to Proposition 6, for any $i_k \in \{1, \dots, n_k\}$ in group $k \in \{1, 2, 3\}$*

$$\begin{bmatrix} \dot{\breve{x}}_1^{i_k}(t) \\ \dot{\breve{x}}_2^{i_k}(t) \\ \dot{\breve{x}}_3^{i_k}(t) \\ \dot{\breve{x}}_4^{i_k}(t) \end{bmatrix} = A_k \begin{bmatrix} \breve{x}_1^{i_k}(t) \\ \breve{x}_2^{i_k}(t) \\ \breve{x}_3^{i_k}(t) \\ \breve{x}_4^{i_k}(t) \end{bmatrix} + B_k \breve{u}^{i_k}(t) \tag{3.49}$$

Similarly, define $\breve{X}_k^{i_k} = [\breve{x}_1^{i_k}(t)\ \breve{x}_2^{i_k}(t)\ \breve{x}_3^{i_k}(t)\ \breve{x}_4^{i_k}(t)]$, for each subgroup $k \in \{1, 2, 3\}$,

$$\dot{\breve{X}}_k^{i_k} = A_k \breve{X}_k^{i_k} + B_k \breve{u}_k^{i_k}(t) \tag{3.50}$$

Proof. The auxiliary model is of following form by subtraction of (3.37) and (3.44):

$$\begin{bmatrix} \dot{\breve{x}}_1^{i_k}(t) \\ \dot{\breve{x}}_2^{i_k}(t) \\ \dot{\breve{x}}_3^{i_k}(t) \\ \dot{\breve{x}}_4^{i_k}(t) \end{bmatrix} = A_k \begin{bmatrix} \breve{x}_1^{i_k}(t) \\ \breve{x}_2^{i_k}(t) \\ \breve{x}_3^{i_k}(t) \\ \breve{x}_4^{i_k}(t) \end{bmatrix} + B_k \breve{u}^{i_k}(t) + C - \bar{C} \tag{3.51}$$

where C defined as a constant in Proposition 2
Thus, $C - \bar{C} = 0$ and (3.49) is obtained.
Furthermore, $x_1^{i_k}$ is the same for all agents of all groups, so that it is identical for all agents in the same group, and the final $\breve{x}_1^{i_k}(t)$ will be 0 all the time. □

For any feedback strategy (L, G), define the following performance index

$$\begin{aligned} J_1^{NH}(L,G) := \lim_{T\to\infty} \Bigg[&\sum_{k=1}^{3} \frac{1}{T} \int_{t=0}^{T} \left(\frac{1}{n_k} \sum_{i_k=1}^{n_k} \alpha_k \Big(\breve{X}_k^{i_k}(t)^\intercal Q \breve{X}_k^{i_k}(t) + R \breve{u}_k^{i_k}(t)^2 \Big) \right) dt \\ &+ \frac{1}{T} \int_{t=0}^{T} \left(\tilde{X}^T \left(\begin{bmatrix} \alpha_1 n_1 Q & 0 & 0 \\ 0 & \alpha_2 n_2 Q & 0 \\ 0 & 0 & \alpha_3 n_3 Q \end{bmatrix} + M^T \tilde{Q} M \right) \tilde{X} \right. \\ &\left. + \begin{bmatrix} \alpha_1 n_1 R & 0 & 0 \\ 0 & \alpha_2 n_2 R & 0 \\ 0 & 0 & \alpha_3 n_3 R \end{bmatrix} \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \\ \bar{u}_3(t) \end{bmatrix}^2 \right) dt \Bigg] \end{aligned} \tag{3.52}$$

Proof. Under Lemma 1, the cost function (3.52) could be expressed as follows:

$$\begin{aligned} J_1^{NH}(L,G) &:= \lim_{T\to\infty} \left[\frac{1}{T} \int_{t=0}^{T} \left(\sum_{k=1}^{3} \sum_{i_k=1}^{n_k} \alpha_k \Big(X^{i_k}(t)^\intercal Q X^{i_k}(t) + R u_k^{i_k}(t)^2 \Big) + \tilde{X}^\intercal \tilde{Q} \tilde{X} \right) dt \right] \\ &= \lim_{T\to\infty} \Bigg[\sum_{k=1}^{3} \frac{1}{T} \int_{t=0}^{T} \Bigg(\sum_{i_k=1}^{n_k} \alpha_k \Big(X_k^{i_k}(t)^\intercal Q X_k^{i_k}(t) + R u_k^{i_k}(t)^2 \Big) \\ &\qquad + \alpha_k n_k \Big(\bar{X}_k(t)^\intercal Q \bar{X}_k(t) + R \bar{u}_k(t)^2 \Big) \Bigg) dt + \frac{1}{T} \int_{t=0}^{T} \left(\tilde{X}^\intercal \tilde{Q} \tilde{X} \right) dt \Bigg] \\ &= \lim_{T\to\infty} \Bigg[\sum_{k=1}^{3} \frac{1}{T} \int_{t=0}^{T} \left(\sum_{i_k=1}^{n_k} \alpha_k \Big(\breve{X}_k^{i_k}(t)^\intercal Q \breve{X}_k^{i_k}(t) + R \breve{u}_k^{i_k}(t)^2 \Big) \right) dt \\ &\qquad + \frac{1}{T} \int_{t=0}^{T} \left(\tilde{X}^\intercal \tilde{Q} \tilde{X} + \sum_{k=1}^{3} \alpha_k n_k \Big(\bar{X}_k(t)^\intercal Q \bar{X}_k(t) + R \bar{u}_k(t)^2 \Big) \right) dt \Bigg] \end{aligned} \tag{3.53}$$

According to Proposition 6, $\bar{X} = \sum_{k=1}^{3} \alpha_k n_k \bar{X}_k$.

Then $\bar{X} = M\tilde{X}$, with
$M =$

$$\begin{bmatrix} \alpha_1 n_1 & 0 & 0 & 0 & \alpha_2 n_2 & 0 & 0 & 0 & \alpha_3 n_3 & 0 & 0 & 0 \\ 0 & \alpha_1 n_1 & 0 & 0 & 0 & \alpha_2 n_2 & 0 & 0 & 0 & \alpha_3 n_3 & 0 & 0 \\ 0 & 0 & \alpha_1 n_1 & 0 & 0 & 0 & \alpha_2 n_2 & 0 & 0 & 0 & \alpha_3 n_3 & 0 \\ 0 & 0 & 0 & \alpha_1 n_1 & 0 & 0 & 0 & \alpha_2 n_2 & 0 & 0 & 0 & \alpha_3 n_3 \end{bmatrix}$$

Here $x_1^{i_k}$ is the same for every single agent, but as $\sum_{k=1}^{3} \alpha_k n_k = 1$, the matrix above is still correct and better balanced.

Thus, the final form of cost function (3.52) is found □

Theorem 2. *Under Assumptions 1 , 3 , 4, the optimal strategy (L^*, G^*) of the auxiliary system of Problem 2 in A_1^{NH} is given as follows:*

$$\breve{u}_k^i(t) = \breve{K}_k \breve{X}_k, \quad \bar{u}_k(t) = G_k \tilde{X}(t) \tag{3.54}$$

optimal gains are given by

$$\begin{aligned} &\breve{K}_k = \text{Feedback}(A_k, B_k, Q_k, R_k) \\ &G = \text{Feedback} \\ &\left(\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} + \begin{bmatrix} D \\ D \\ D \end{bmatrix}, \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix}, \right. \\ &\left. \begin{bmatrix} \alpha_1 n_1 & 0 & 0 \\ 0 & \alpha_2 n_2 & 0 \\ 0 & 0 & \alpha_3 n_3 \end{bmatrix} Q + M^T \tilde{Q} M, \begin{bmatrix} \alpha_1 n_1 R & 0 & 0 \\ 0 & \alpha_2 n_2 R & 0 \\ 0 & 0 & \alpha_3 n_3 R \end{bmatrix} \right). \\ &G_1 = G(1,:), \quad G_2 = G(2,:), \quad G_3 = G(3,:) \end{aligned} \tag{3.55}$$

Thus, optimal gains are as follows:

$$
\begin{aligned}
&[L^i_{f_1}, L^i_{p_1}, L^i_{x_1}] = \breve{K}_1(:, 2:4)\\
&G_{i_1} = \tilde{G}_1(1) + \tilde{G}_1(5) + \tilde{G}_1(9)\\
&[K_{f_{11}}, K_{p_{11}}, K_{x_{11}}] = \tilde{G}_1(:, 2:4) - \breve{K}_1(:, 2:4)\\
&[K_{f_{12}}, K_{p_{12}}, K_{x_{12}}] = \tilde{G}_1(:, 6:8)\\
&[K_{f_{13}}, K_{p_{13}}, K_{x_{13}}] = \tilde{G}_1(:, 10:12)\\
&[L^i_{f_2}, L^i_{p_2}, L^i_{x_2}] = \breve{K}_2(:, 2:4)\\
&G_{i_2} = \tilde{G}_2(1) + \tilde{G}_2(5) + \tilde{G}_2(9)\\
&[K_{f_{21}}, K_{p_{21}}, K_{x_{21}}] = \tilde{G}_2(:, 2:4)\\
&[K_{f_{22}}, K_{p_{22}}, K_{x_{22}}] = \tilde{G}_2(:, 6:8) - \breve{K}_2(:, 2:4)\\
&[K_{f_{23}}, K_{p_{23}}, K_{x_{23}}] = \tilde{G}_2(:, 10:12)\\
&[L^i_{f_3}, L^i_{p_3}, L^i_{x_3}] = \breve{K}_3(:, 2:4)\\
&G_{i_3} = \tilde{G}_3(1) + \tilde{G}_3(5) + \tilde{G}_3(9)\\
&[K_{f_{31}}, K_{p_{31}}, K_{x_{31}}] = \tilde{G}_3(:, 2:4)\\
&[K_{f_{32}}, K_{p_{32}}, K_{x_{32}}] = \tilde{G}_3(:, 6:8)\\
&[K_{f_{33}}, K_{p_{33}}, K_{x_{33}}] = \tilde{G}_3(:, 10:12) - \breve{K}_3(:, 2:4)
\end{aligned}
\tag{3.56}
$$

Proof. Same as in Theorem 1 □

3.4.3 Numerical Results for Problem 2

In this section, we present some numerical results for typical values of the system parameters extracted from [Tan, 2009, 2011].

Consider 3 homogeneous population of n_1, n_2, n_3 generators, where $n_1 + n_2 + n_3 = N$ and the distribution of initial difference of frequency Δf_k is uniform between $0 \sim 0.08$Hz, $-0.06 \sim -0.02$Hz, $-0.08 \sim -0.06$ Hz, respectively(so that the total summation of frequency deviation won't pass 0.15 p.u.Hz); $\Delta P^i_{g_k}$ and $\Delta X^i_{g_k}$ are also uniformly distributed around 0 between -0.02/n and 0.02/n p.u.MW(because the per unit base used here are the areawise base so that it will be separate equally for each individual).

The load disturbances imparted in the simulation and the chose of $p_f, q_f, \tilde{p}$ are as in Section 3.3.3.

Parameters of each homogeneous group, taken from [Tan, 2009], are as follows:

$$
\begin{aligned}
&K_{P1} = 120 \quad T_{P1} = 20 \quad T_{T1} = 0.3 \quad T_{G1} = 0.08 \\
&K_{P2} = 115 \quad T_{P2} = 15 \quad T_{T2} = 0.375 \quad T_{G2} = 0.085 \\
&K_{P3} = 112.5 \quad T_{P3} = 25 \quad T_{T3} = 0.33 \quad T_{G3} = 0.072
\end{aligned}
\tag{3.57}
$$

So that:

$$
\begin{aligned}
G_{g1} &= \frac{1}{0.08s+1} \quad G_{t1} = \frac{1}{0.3s+1} \quad G_{p1} = \frac{120}{20s+1} \\
G_{g2} &= \frac{1}{0.085s+1} \quad G_{t2} = \frac{1}{0.375s+1} \quad G_{p2} = \frac{115}{15s+1} \\
G_{g3} &= \frac{1}{0.075s+1} \quad G_{t2} = \frac{1}{0.33s+1} \quad G_{p3} = \frac{112.5}{25s+1}
\end{aligned}
\tag{3.58}
$$

120 systems, correspond to 3 different types of generators, 60 medium size 1500MW, 40 large 2000MW, 20 small 1200 MW(so that $J_1 = 1.5$, $J_2 = 2$, $J_3 = 1.2$ are inversely proportional to the time constants T_{pk}), are taken into simulation and with the objective of having their mean frequency deviations from nominal converge to zero $\Delta\bar{f}_{refk} = 0, k \in \{1, 2, 3\}$. As shown in Figure 3.7, Output trajectories from 10 randomly selected machines within each of the homogeneous subgroups are plotted together on each graph, with the average mean of each subgroup respectively marked red, blue, green. Also, the total frequency tracking error for each subgroup, as well as the global frequency deviations, marked black, are plotted on a separate graph.

Same as in previous Section 3.3.3, a disturbance hit the initial generator system and make initial frequency deviations different one from each other. We plot 3 different homogeneous group separately, starting from around 0.05, -0.05 and 0 Hz, respectively. This time, instead of homogeneous case, similar load changes are added in similar time but the way of distribution(according to its kinetic value), presented in Section 3.2.3, is different.

A total disturbance P_d of 0.03, and 0.02 p.u.MW(because of areawise per unit base, these value will be very small even if it's big in MW) are added to all agents at $\frac{T}{4}$ and $\frac{T}{3}$, so that each individual in group 1 is disturbed with $\alpha_1 P_d$, that in group 2 with $\alpha_2 P_d$ and that in group 2 with $\alpha_3 P_d$.

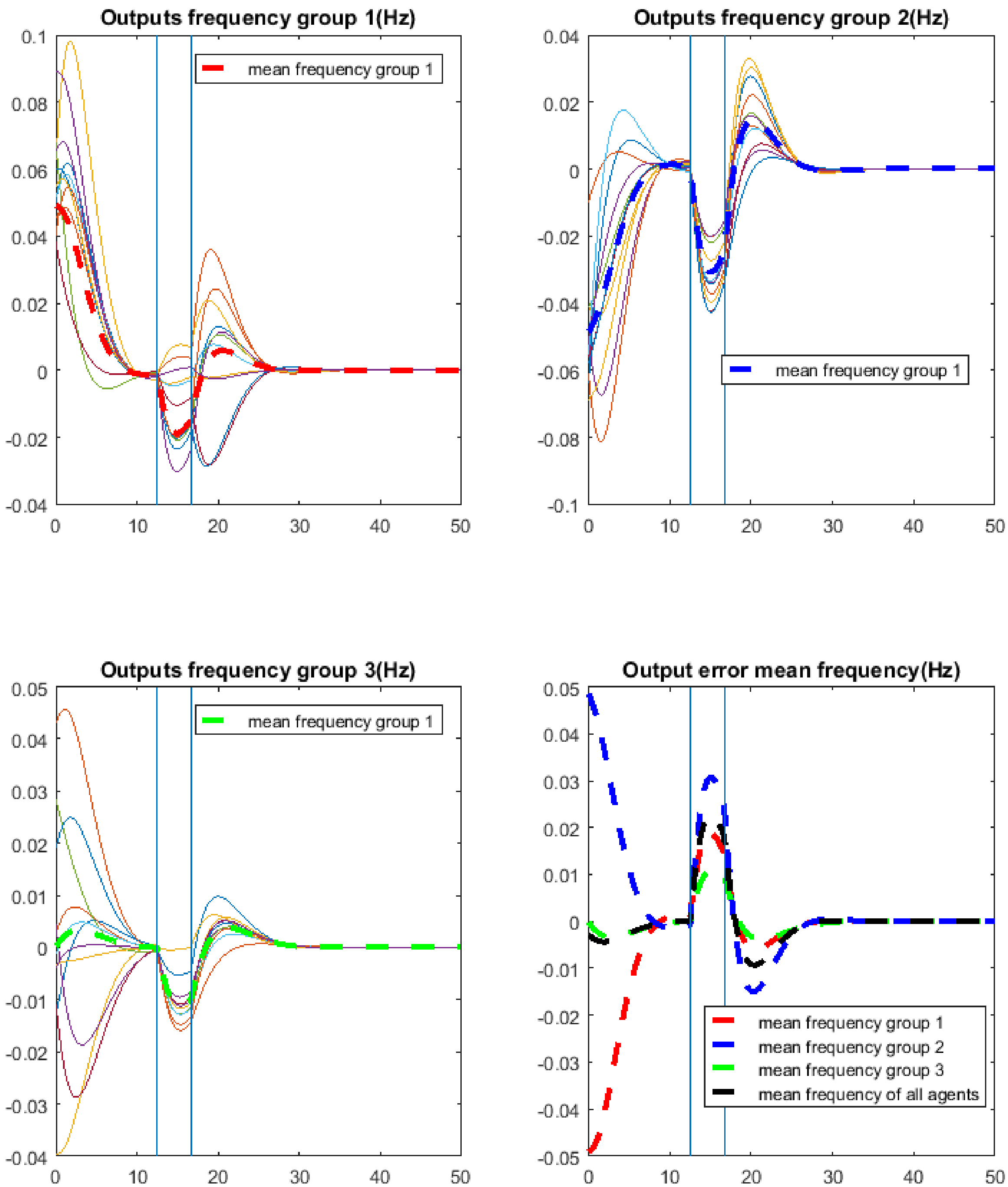

Figure 3.7 The plot of 30 randomly selected non homogeneous agents of single area control systems under MFS-IS.

In Figure 3.7, we can observe a behaviour similar to that in Section 3.3.3 in each homogeneous group, where each agent frequency is driven towards the average and the average itself converges to the preset target. Again, this is an advantage when compared to LQR applied to macromachine representations. It is due to the possibility of local controls contributed by team theory. The only difference relative to the homogeneous case is that, in this case, each homogeneous subgroup will have a different model specific local controller, and its contribution to overall mean frequency will be weighted by the total inertia value of the subgroup. Thus the 3 subgroups contribute differently, but the global average still ultimately achieve zero deviation from target. This confirms the effectiveness of team theory controllers in the non-homogeneous case.

According to the graph, we could find that the frequency deviation are almost proposition to the inertia value, as group 2 is most disturbed, followed by group 1 then group 3. In graph "Output error mean frequency" (right below) we could also observe that the total average is influenced not only by inertia value but also the number of agents, so that it is between group 2 and group 1. Moreover, group 3, which has the least inertia value, has very small number thus has very small weight.

CHAPTER 4 2 AREA LFC

4.1 Problem Formulation

4.1.1 Notation

Table 4.1 Notation for 2 area LFC

Symbol	Description
K_P	Electric system gain(Hz/p.u.MW)
T_P	Electric system time constant (s)
T_T	Turbine time constant (s)
T_G	Governor time constant (s)
$\Delta f(t)$	Incremental frequency deviation (Hz)
$\Delta P_g(t)$	Incremental change in generator output (p.u.MW)
$\Delta X_g(t)$	Incremental change in governor value position(p.u.MW)
ΔP_d	Load disturbance (p.u.MW)
ΔP_{tie}	Tie line power between areas(p.u.MW)
ACE	Area control error for each area(p.u.MW)
a_{12}	Proportional parameter between tie-line deviation
T_{12}	Tie-line parameter between area 1 and 2(p.u.MW)
u	Incremental change in the speed changer position(p.u.MW)

Same as in Section 3.1.1, new concept area control error will be discussed in future section.

4.1.2 Preliminary Discussion of ACE

When dealing with a single area, the paramount control objective is to maintain each machine frequency close to its nominal value, and in particular the area wide frequency (weighted empirical average of individual machine frequencies) at its nominal value. This is why the objective function to be minimized by machines acting as a team include square deviations of individual machine frequencies from nominal values, as well as both the square deviation of weighted average frequency from nominal value and the square of the integral of that deviation, *hence reflecting a PI control view within a linear quadratic optimal control framework.* In the case of two areas, the control objectives become multiple in that the frequency objectives of the single area problems must still be reflected, but also, each area has to strive to maintain the tie line interchange power close to its scheduled value. The traditional approach to handle this multiple objective situation has been to define for each area a so-called area control error signal $ACE_k := \beta_k \Delta f_k + \Delta P_{tie_k}$, $k = 1, 2$, where herein k refers to the area number, and β_k's are crucial coefficients reflecting not only the relative priority of frequency and tie line objectives, but also leaving the possibility of a given area accepting temporarily to deviate from its desired mean frequency in order to temporarily help another area in need (see the discussion in [Kundur et al., 1994]).

In our formulation, we shall maintain the spirit of this classical time tried control philosophy, by introducing a new ACE related state in the area state representations. Also, we shall introduce the integral of ACE as an extra state, *in keeping with the PI control approach.*

4.1.3 Information Structure

The information structure of each individual is nearly the same as in the single area homogeneous case of Section 3.1.2. The difference is that besides every agent observing its own frequency deviation, the individual agents also observe the current tie line power interchange, based on which they can estimate their own contribution to this power interchange, namely $\frac{1}{n_k}\Delta P_{tie_k}$, and their own contribution to the global ACE signal in the area denoted $ACE_k^i := \beta_k \Delta f_k^i + \frac{1}{n_k}\Delta P_{tie_k}$, while $k = \{1, 2\}$ for the i^{th} agent in area k. Also, the global ACE's in the two areas are assumed to be observed by agents in both areas.

We propose two different approaches for the 2 area homogeneous LFC problem. At first we introduce *a team solution*, in Section 4.4, where we consider these 2 areas as a large team sharing the same overall cost function so that the mean field of each area will be shared with the other area to compute the optimal solution, including ACE_k, $\int ACE_k$, $\Delta\bar{f}_k$, $\Delta\bar{P}_{gk}$ and $\Delta\bar{X}_{gk}$, where $k \in \{1, 2\}$ denote different areas.

Then, recognizing that the areas are typically operated by independent power companies which will proritize their own areas, we consider *a game theoretic solution*, in Section 4.5, with each area represented for simplicity as a single aggregate machine. The Nash equilibrium policies when they exist, will in general be feedback laws on the joint state of the two areas. However, if we neglect the feedback gains on non area state variables, we obtain decoupled policies as in power systems current LFC implementations. We numerically compare the behaviors of the fully coupled and areawise decoupled control policies.

Although the two setups (team setup and game setup) are distinct in terms of control philosophy, they share a similar control block diagram under the same information structure, namely the mean-field sharing information structure which has already been detailed in Section 3.1.2, of the 2 areas with homogeneous machines denoted A_2^H depicted in Figure 4.3 (further below).

4.1.4 The 2 Area Optimization Problems: Problem 3 and Problem 4

Optimization Problem 3: The Two Areas Team Optimal Case

Here we define the following additive performance index for two area team optimal control problem:

$$
\begin{aligned}
J_{21}^{HT}(L_1, G_1) &:= \lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T}\Bigg[\frac{1}{n_1}\sum_{i=1}^{n_1}\Big[q_1(ACE_{ref_1}^i(t) - ACE_1^i(t))^2 + R_1 u_1^i(t)^2\Big] \\
&\qquad + q_{I1}(\int_{\tau=0}^{t} A\bar{C}E_{ref_1}(\tau) - A\bar{C}E_1(\tau)d\tau)^2\Bigg]dt \\
J_{22}^{HT}(L_2, G_2) &:= \lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T}\Bigg[\frac{1}{n_2}\sum_{i=1}^{n_2}\Big[q_2(ACE_{ref_2}^i(t) - ACE_2^i(t))^2 + R_2 u_2^i(t)^2\Big] \\
&\qquad + q_{I2}(\int_{\tau=0}^{t} A\bar{C}E_{ref_2}(\tau) - A\bar{C}E_2(\tau)d\tau)^2\Bigg]dt
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
J_2^{HT}(L, G) &:= J_{21}^{HT}(L_1, G_1) + J_{22}^{HT}(L_2, G_2) \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T}\Bigg[q_{I1}(\int_{\tau=0}^{t} A\bar{C}E_{ref_1} - A\bar{C}E_1(\tau)d\tau)^2 \\
&\qquad + q_{I2}(\int_{\tau=0}^{t} A\bar{C}E_{ref_2} - A\bar{C}E_2(\tau)d\tau)^2 \\
&\qquad + \frac{1}{n_1}\sum_{i=1}^{n_1}\Big[q_1(ACE_{ref_1}^i(t) - ACE_1^i(t))^2 + R_1 u_1^i(t)^2\Big] \\
&\qquad + \frac{1}{n_2}\sum_{i=1}^{n_2}\Big[q_2(ACE_{ref_2}^i(t) - ACE_2^i(t))^2 + R_2 u_2^i(t)^2\Big]\Bigg]dt
\end{aligned}
\tag{4.2}
$$

Problem 3. *The objective is to minimize the cost function J_2^{HT} under mean-field sharing information structure such that:*

$$
J_2^{HT*} := J_2^{HT}(L^*, G^*) \leq J_2^{HT}(L, G). \tag{4.3}
$$

L, G should be independent of the choice of reference trajectories.

Optimization Problem 4: The Two Areas Game Nash Equilibrium Case

For the two areas game, we define the following performance functions:

$$
\begin{aligned}
J_{21}^{HG}(L_1, G_1, G_2) := \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\Bigg[\frac{1}{n_1}\sum_{i=1}^{n_1}\Big[q_1(ACE_{ref_1}^i(t) - ACE_1^i(t))^2 + R_1 u_1^i(t)^2\Big] \\
+ q_{12}(A\bar{C}E_{ref_2} - A\bar{C}E_2(t))^2 \\
+ q_{I1}(\int_{\tau=0}^{t}(A\bar{C}E_{ref_1} - A\bar{C}E_1(\tau))d\tau)^2 \\
+ q_{I12}(\int_{\tau=0}^{t}(A\bar{C}E_{ref_2} - A\bar{C}E_2(\tau))d\tau)^2\Bigg]dt \\
J_{22}^{HG}(L_2, G_1, G_2) := \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\Bigg[\frac{1}{n_2}\sum_{i=1}^{n_2}\Big[q_2(ACE_{ref_2}^i(t) - ACE_2^i(t))^2 + R_2 u_2^i(t)^2\Big] \\
+ q_{21}(A\bar{C}E_{ref_1} - A\bar{C}E_1(t))^2 \\
+ q_{I2}(\int_{\tau=0}^{t}(A\bar{C}E_{ref_2} - A\bar{C}E_2(\tau))d\tau)^2 \\
+ q_{I21}(\int_{\tau=0}^{t}(A\bar{C}E_{ref_1} - A\bar{C}E_1(\tau))d\tau)^2\Bigg]dt
\end{aligned}
\tag{4.4}
$$

Problem 4. *Here, the objective is to achieve a Nash equilibrium if it exists given the cost functions* J_{21}^{HG} *and* J_{22}^{HG} *under mean-field sharing information structure such that:*

$$
\begin{aligned}
J_{21}^{HG*} :=& J_{21}^{HG}(L_1^*, G_1^*, G_2^*) \leq J_{21}^{HG}(L_1, G_1, G_2^*) \\
J_{22}^{HG*} :=& J_{21}^{HG}(L_2^*, G_1^*, G_2^*) \leq J_{21}^{HG}(L_2, G_1^*, G_2).
\end{aligned}
\tag{4.5}
$$

$L_1, G_1,\ L_2, G_2$ *should be independent of the choice of reference trajectories.*

4.1.5 Comparison of Problem 3 and Problem 4

In a forthcoming section, we shall contrast the results of the two set-ups as specified in the table below:

Table 4.2 Comparison of Team Optimum Performance and Game Nash Equilibrium Performance

Case	Team optimum	Nash equilibrium
Theory	Mean-field team	Mean-field game
Cost function	equation(4.2)	equation(4.4)
Objective	Optimum for total system	Nash equilibrium
Use	run by same company	run by different companies
Section	Section 4.4	Section 4.5

4.2 Generator Modelling for Two Area System

4.2.1 Aggregate Generator Model

A macromachine model in one of two interconnected areas is shown in Figure 4.1, where all the variables above are defined in Table 4.1 and the plant for load frequency control consists of 5 parts:

— Governor with dynamics: $G_g = \frac{1}{T_G s+1}$
— Turbine generator with dynamics:$G_t = \frac{1}{T_T s+1}$
— Power systems with dynamics:$G_p = \frac{K_P}{T_P s+1}$
— Disturbance ΔP_d
— Tie line power ΔP_{tie}

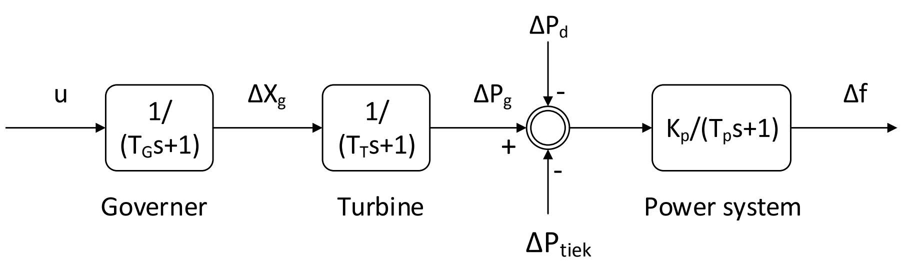

Figure 4.1 Linear model of generator in 2 areas case

So that the new equation is the following, for area $k \in \{1, 2\}$:

$$\begin{aligned}
\Delta X_{g_k}(t) &= G_{g_k} u(t) \\
\Delta P_{g_k}(t) &= G_{t_k} \Delta X_{g_k}(t) \\
\Delta f_k(t) &= G_{p_k}(\Delta P_{g_k}(t) - \Delta P_{d_k}(t) - \Delta P_{tie_k})
\end{aligned} \tag{4.6}$$

which can be represented as follows with ΔP_{tie_k} as a special input or disturbance for areas $k = 1, 2$:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\Delta f_1(t)\\ \Delta P_{g_1}(t)\\ \Delta X_{g_1}(t)\end{bmatrix} &= \begin{bmatrix}-\frac{1}{T_{P_1}} & \frac{K_{P1}}{T_{P_1}} & 0\\ 0 & -\frac{1}{T_{T_1}} & \frac{1}{T_{T_1}}\\ 0 & 0 & -\frac{1}{T_{G_1}}\end{bmatrix}\begin{bmatrix}\Delta f_1(t)\\ \Delta P_{g_1}(t)\\ \Delta X_{g_1}(t)\end{bmatrix}\\ &+ \begin{bmatrix}0\\ 0\\ \frac{1}{T_{G_1}}\end{bmatrix}u - \begin{bmatrix}\frac{K_{P_1}}{T_{P_1}}\\ 0\\ 0\end{bmatrix}\Delta P_{tie_1}\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\Delta f_2(t)\\ \Delta P_{g_2}(t)\\ \Delta X_{g_2}(t)\end{bmatrix} &= \begin{bmatrix}-\frac{1}{T_{P_2}} & \frac{K_{P_2}}{T_{P_2}} & 0\\ 0 & -\frac{1}{T_{T_2}} & \frac{1}{T_{T_2}}\\ 0 & 0 & -\frac{1}{T_{G_2}}\end{bmatrix}\begin{bmatrix}\Delta f_2(t)\\ \Delta P_{g_2}(t)\\ \Delta X_{g_2}(t)\end{bmatrix}\\ &+ \begin{bmatrix}0\\ 0\\ \frac{1}{T_{G_2}}\end{bmatrix}u - \begin{bmatrix}\frac{K_{P_2}}{T_{P_2}}\\ 0\\ 0\end{bmatrix}\Delta P_{tie_2}\end{aligned} \tag{4.8}$$

Proof. Same as in Section 3.2 □

4.2.2 Homogeneous Multi Machines Modelling in Two Area Case

Same as in Section 3.2.2, the result is represented in Figure 4.2

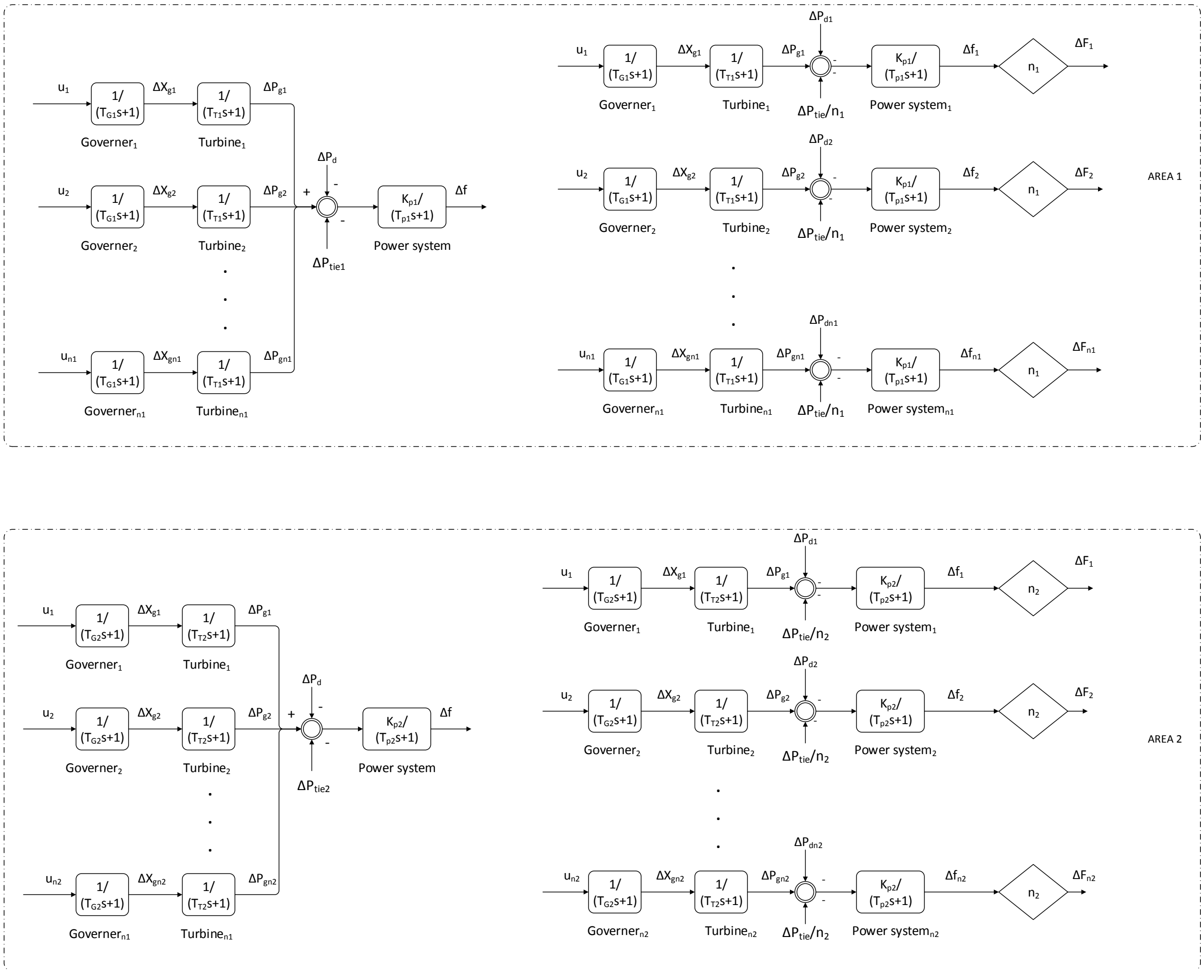

Figure 4.2 Generator modelling in 2 area homogeneous case

The modelling principle (dividing total area generation and load into individualized inputs proportional to the size of machines) is the same as in the earlier cases, but given the presence of two areas and a tie line power interchange between them, when the macromachines are divided into their individual building blocks in each area, the associated state space space models will involve variables from the other area. This is why, even when studying the team optimal formulation of the two area system, we "pretend" we are dealing with two non homogeneous groups of machines with state components labelled 1 and 2 respectively for areas 1 and 2. Furthermore, in order to remain in line with the team optimal formalism of [Arabneydi, 2016] , we then limit the information exchanges between machines in different areas to only global (i.e. based on areawise state space component averages) state variables which are defined in equation (4.17) below. Finally, note that we need to with the same per unit bases for both areas. For simplicity, we shall assume that both areas each hold equal generation capacity so that the per unit power base will be chosen as a single area total capacity.

4.3 State-space Representation and Control Block Diagram for A_2^H

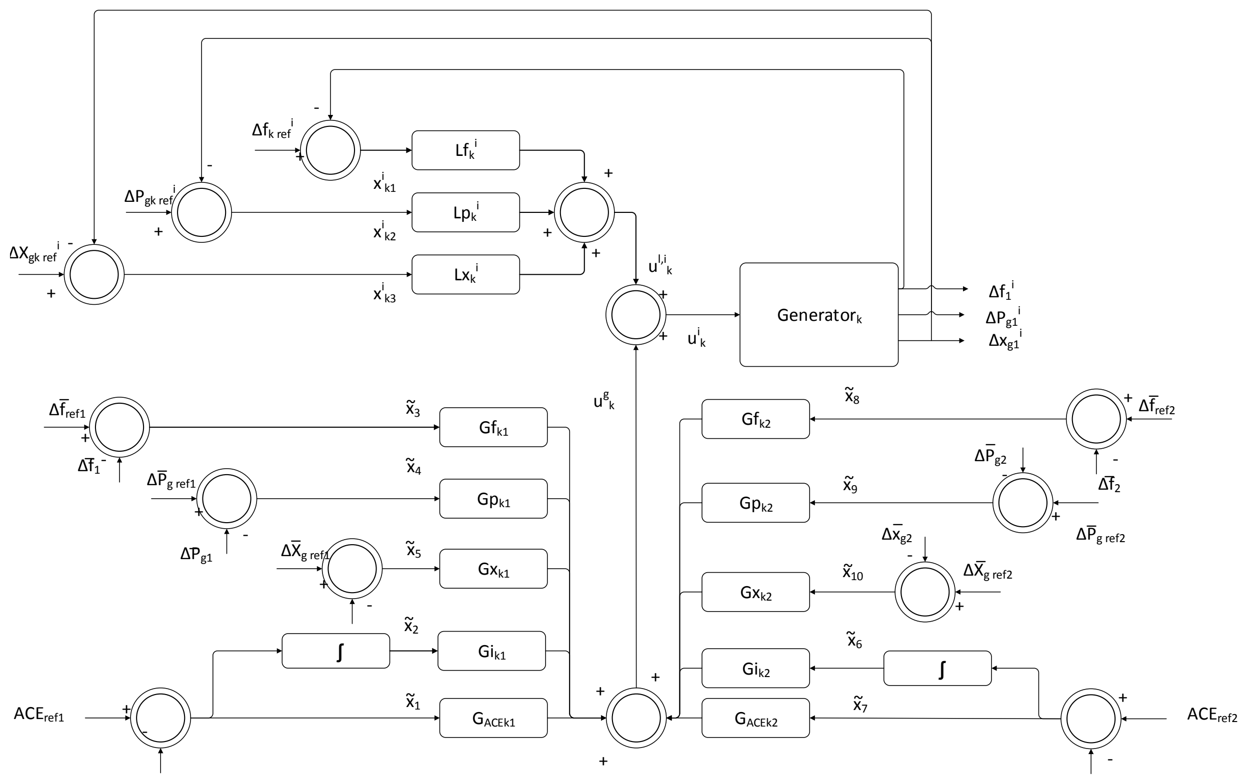

Figure 4.3 The control block diagram under MFS-IS for 2 area system $i \in \{1, \ldots, n\}$, where the optimal gains are given by Theorem 3(The Team Optimal Control Method) or Theorem 4(The Game Theoretic View).

Following [Das et al., 2013] and [Arabneydi, 2016], we construct a state-space representation for Problem 3 and Problem 4.

Proposition 10. *Let Assumption 1 hold.*
There exists a state-space representation for local controller as follows, for both area $k \in \{1, 2\}$.

$$\begin{bmatrix} \dot{x}^i_{k1}(t) \\ \dot{x}^i_{k2}(t) \\ \dot{x}^i_{k3}(t) \end{bmatrix} = A_k \begin{bmatrix} x^i_{k1}(t) \\ x^i_{k2}(t) \\ x^i_{k3}(t) \end{bmatrix} + B_k u^i_k(t) + C_k + D_k \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \\ \tilde{x}_5(t) \\ \tilde{x}_6(t) \\ \tilde{x}_7(t) \\ \tilde{x}_8(t) \\ \tilde{x}_9(t) \\ \tilde{x}_{10}(t) \end{bmatrix} \tag{4.9}$$

where x^i_{kj}, $j \in \{1, 2, 3\}$, $k \in \{1, 2\}$ *denotes the difference between* Δf^i_k, ΔP^i_{gk}, ΔX^i_{gk} *and their references in area* k, *all global variables* $[\tilde{x}_1 ... \tilde{x}_{10}]$ *defined below in* (4.17) *and*

$$A_k := \begin{bmatrix} -\frac{1}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \\ 0 & -\frac{1}{T_{T_k}} & \frac{1}{T_{T_k}} \\ 0 & 0 & -\frac{1}{T_{G_k}} \end{bmatrix}, B_k := \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_{G_k}} \end{bmatrix},$$

$$C_k := A \begin{bmatrix} 0 \\ \Delta P^i_{d_k}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{K_{P_k}}{T_{P_k}} \Delta P^i_{d_k}(t) \\ -\frac{1}{T_{T_k}} \Delta P^i_{d_k}(t) \\ 0 \end{bmatrix} \tag{4.10}$$

$$D_1 := \begin{bmatrix} 0 & \frac{K_{P_1}}{T_{P_1}} & -\beta_1 \frac{K_{P_1}}{T_{P_1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$D_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_{P_2}}{T_{P_2}} & -\beta_2 \frac{K_{P_2}}{T_{P_2}} & 0 & 0 \end{bmatrix}^T,$$

Proof. Same as in Proposition 2, it follows from (4.7), (4.8) and Assumption 1 that

$$\begin{bmatrix} \dot{x}^i_{k1}(t) \\ \dot{x}^i_{k2}(t) \\ \dot{x}^i_{k3}(t) \end{bmatrix} = A_k \begin{bmatrix} x^i_{k1}(t) \\ x^i_{k2}(t) \\ x^i_{k3}(t) \end{bmatrix} + B_k u^i_k(t) + C_k + D_k \frac{1}{n_k} \Delta P_{tie1} \tag{4.11}$$

where the derivatives of the reference trajectory are zero because they are assumed to be

constant according to Assumption 1.
Besides, according to our definitions: $\frac{1}{n_k}\Delta P_{tie_k} = A\bar{C}E_k - \beta_k \Delta \bar{f}_k$, Thus,

$$\begin{bmatrix} \dot{x}^i_{k1}(t) \\ \dot{x}^i_{k2}(t) \\ \dot{x}^i_{k3}(t) \end{bmatrix} = A_k \begin{bmatrix} x^i_{k1}(t) \\ x^i_{k2}(t) \\ x^i_{k3}(t) \end{bmatrix} + B_k u^i_k(t) + C_k + D_k \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \\ \tilde{x}_5(t) \\ \tilde{x}_6(t) \\ \tilde{x}_7(t) \\ \tilde{x}_8(t) \\ \tilde{x}_9(t) \\ \tilde{x}_{10}(t) \end{bmatrix} \tag{4.12}$$

$\square$

Proposition 11. *Let Assumption 1 hold.*
We define an auxiliary system for both area $k \in \{1, 2\}$ where the influence of ΔP_{tiek} can be annulled:

$$\begin{bmatrix} \dot{\breve{x}}^i_{k1}(t) \\ \dot{\breve{x}}^i_{k2}(t) \\ \dot{\breve{x}}^i_{k3}(t) \end{bmatrix} = A_k \begin{bmatrix} \breve{x}^i_{k1}(t) \\ \breve{x}^i_{k2}(t) \\ \breve{x}^i_{k3}(t) \end{bmatrix} + B_k \breve{u}^i_k(t) \tag{4.13}$$

where $\breve{x}^i_{kj}$, $j \in \{1, 2, 3\}$, $k \in \{1, 2\}$ marks the difference between Δf^i_k, ΔP^i_{gk}, ΔX^i_{gk} and their average in area k, $\breve{u}^i_k = u^i_k - \bar{u}^i_k$ and

$$A_k := \begin{bmatrix} -\frac{1}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \\ 0 & -\frac{1}{T_{T_k}} & \frac{1}{T_{T_k}} \\ 0 & 0 & -\frac{1}{T_{G_k}} \end{bmatrix}, B_k := \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_{G_k}} \end{bmatrix}, \tag{4.14}$$

Proof. By averaging of (4.9) over all systems in its area, we get:

$$\begin{bmatrix} \dot{\bar{x}}_{k1}(t) \\ \dot{\bar{x}}_{k2}(t) \\ \dot{\bar{x}}_{k3}(t) \end{bmatrix} = A_k \begin{bmatrix} \bar{x}_{k1}(t) \\ \bar{x}_{k2}(t) \\ \bar{x}_{k3}(t) \end{bmatrix} + B_k \bar{u}_k(t) + \bar{C}_k + D_k \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \\ \tilde{x}_5(t) \\ \tilde{x}_6(t) \\ \tilde{x}_7(t) \\ \tilde{x}_8(t) \\ \tilde{x}_9(t) \\ \tilde{x}_{10}(t) \end{bmatrix} \tag{4.15}$$

Then as ΔP_{tie_k} is divided equally among all agents, (4.13) is obtained by simple subtraction of (4.9) and (4.15) □

Proposition 12. *Let Assumption 1 hold. There exists a state-space representation for global controller as follows.*

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \\ \dot{\tilde{x}}_3(t) \\ \dot{\tilde{x}}_4(t) \\ \dot{\tilde{x}}_5(t) \\ \dot{\tilde{x}}_6(t) \\ \dot{\tilde{x}}_7(t) \\ \dot{\tilde{x}}_8(t) \\ \dot{\tilde{x}}_9(t) \\ \tilde{x}_{10}(t) \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \\ \tilde{x}_5(t) \\ \tilde{x}_6(t) \\ \tilde{x}_7(t) \\ \tilde{x}_8(t) \\ \tilde{x}_9(t) \\ \tilde{x}_{10}(t) \end{bmatrix} + \tilde{B} \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & \tilde{C}_2 \end{bmatrix} \tag{4.16}$$

where parameters global variables defined below:

$$\begin{aligned}
\tilde{x}_1 &= \int_{\tau=0}^{t} (A\bar{C}E_{ref_1} - A\bar{C}E_1(\tau))d\tau \\
\tilde{x}_2 &= A\bar{C}E_{ref_1} - A\bar{C}E_1(t) \\
\tilde{x}_3 &= \Delta\bar{f}_{ref_1} - \Delta\bar{f}_1 \\
\tilde{x}_4 &= \Delta\bar{P}_{gref_1} - \Delta\bar{P}_{g_1} \\
\tilde{x}_5 &= \Delta\bar{X}_{gref_1} - \Delta\bar{X}_{g_1} \\
\tilde{x}_6 &= \int_{\tau=0}^{t} (A\bar{C}E_{ref_2} - A\bar{C}E_2(\tau))d\tau \\
\tilde{x}_7 &= A\bar{C}E_{ref_2} - A\bar{C}E_2(t) \\
\tilde{x}_8 &= \Delta\bar{f}_{ref_2} - \Delta\bar{f}_2 \\
\tilde{x}_9 &= \Delta\bar{P}_{gref_2} - \Delta\bar{P}_{g_2} \\
\tilde{x}_{10} &= \Delta\bar{X}_{gref_2} - \Delta\bar{X}_{g_2}
\end{aligned} \tag{4.17}$$

and

$$A(1:5,:) :=$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -\beta_1 \frac{K_{P1}}{T_{P1}} & \frac{\beta_1^2 K_{P1}}{T_{P1}} - \beta_1 \frac{1}{T_{P1}} + T_{12} & \beta_1 \frac{K_{P1}}{T_{P1}} & 0 \\ 0 & -\frac{K_{P1}}{T_{P1}} & \frac{\beta_1 K_{P1}}{T_{P1}} - \frac{1}{T_{P1}} & \frac{K_{P1}}{T_{P1}} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_{T1}} & \frac{1}{T_{T1}} \\ 0 & 0 & 0 & 0 & -\frac{1}{T_{G1}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -T_{12}\frac{n_2}{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A(6:10,:) :=$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -T_{12}\frac{n_1}{n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\beta_2 \frac{K_{P2}}{T_{P2}} & \frac{\beta_2^2 K_{P2}}{T_{P2}} - \beta_2 \frac{1}{T_{P2}} + T_{12} & \beta_2 \frac{K_{P2}}{T_{P2}} & 0 \\ 0 & \frac{K_{P2}}{T_{P2}} & \frac{\beta_2}{T_{P2}} - \frac{1}{T_{P2}} & \frac{K_{P2}}{T_{P2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_{T2}} & \frac{1}{T_{T2}} \\ 0 & 0 & 0 & 0 & -\frac{1}{T_{G2}} \end{bmatrix} \tag{4.18}$$

$$B := \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{T_{G1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_{G2}} \end{bmatrix}^T$$

$$\tilde{C}_1 := \begin{bmatrix} 0 \\ \beta_1 \frac{K_{P_1}}{T_{P_1}} \Delta P_{d_1}^i(t) \\ \frac{K_{P_1}}{T_{P_1}} \Delta P_{d_1}^i(t) \\ -\frac{1}{T_{T_1}} \Delta P_{d_1}^i(t) \\ 0 \end{bmatrix} \quad \tilde{C}_2 := \begin{bmatrix} 0 \\ \beta_2 \frac{K_{P_2}}{T_{P_2}} \Delta P_{d_2}^i(t) \\ \frac{K_{P2}}{T_{P_2}} \Delta P_{d_2}^i(t) \\ -\frac{1}{T_{T_2}} \Delta P_{d_2}^i(t) \\ 0 \end{bmatrix}$$

Proof. From definition of ACE we get:

$$\begin{aligned} A\bar{C}E_1 &= \beta_1 \Delta \bar{f}_1 + \frac{1}{n_1}\Delta P_{tie_1} \\ A\bar{C}E_2 &= \beta_2 \Delta \bar{f}_2 + \frac{1}{n_2}\Delta P_{tie_2} \end{aligned} \tag{4.19}$$

Then

$$\begin{aligned} \Delta\dot{\bar{f}}_k &= \begin{bmatrix} -\frac{1}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\bar{f}_k(t) \\ \Delta\bar{P}_{g_k}(t) \\ \Delta\bar{X}_{g_k}(t) \end{bmatrix} - \frac{K_{P_k}}{T_{P_k}}\frac{1}{n_k}\Delta P_{tie_k} \\ &= \begin{bmatrix} -\frac{1}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\bar{f}_k(t) \\ \Delta\bar{P}_{g_k}(t) \\ \Delta\bar{X}_{g_k}(t) \end{bmatrix} - \frac{K_{P_k}}{T_{P_k}}(A\bar{C}E_k - \beta_k\Delta\bar{f}_k) \\ &= \begin{bmatrix} -\frac{1}{T_{P_k}} + \beta_k\frac{K_{P_k}}{T_{P_k}} & \frac{K_{P_k}}{T_{P_k}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\bar{f}_k(t) \\ \Delta\bar{P}_{g_k}(t) \\ \Delta\bar{X}_{g_k}(t) \end{bmatrix} - \frac{K_{P_k}}{T_{P_k}}A\bar{C}E_k \\ A\dot{\bar{C}}E_k &= \frac{d}{dt}(\beta_k\Delta\bar{f}_k + \frac{1}{n_k}\Delta P_{tie_k}) = \beta_k\Delta\dot{\bar{f}}_k + \frac{T_{12}}{n_k}(n_k\Delta\bar{f}_k - n_{(3-k)}\Delta\bar{f}_{(3-k)}) \\ &= \begin{bmatrix} T_{12} - \frac{\beta_k}{T_{P_k}} + \beta_k^2\frac{K_{P_k}}{T_{P_k}} & \beta_k\frac{K_{P_k}}{T_{P_k}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\bar{f}_k(t) \\ \Delta\bar{P}_{g_k}(t) \\ \Delta\bar{X}_{g_k}(t) \end{bmatrix} \\ &\qquad - \beta_k\frac{K_{P_k}}{T_{P_k}}A\bar{C}E_k - T_{12}\frac{n_{(3-k)}}{n_k}\Delta\bar{f}_{(3-k)} \end{aligned} \tag{4.20}$$

Finally, by adding these together, we get (4.16), where the derivatives of the reference trajectory are zero because they is assumed to be constant from Assumption 1.

□

The control action of generic generator i at time t in area k can be expressed in terms of the state-space representations in Propositions 10 and 12 and the block diagram in Figure 4.3, i.e.,

$$\begin{aligned} u_k^i(t) = {} & L_{f_k}^i x_{k1}^i(t) + L_{p_k}^i x_{k2}^i(t) + L_{x_k}^i x_{k3}^i(t) \\ & + G_{I_{k1}}\tilde{x}_1(t) + G_{ACE_{k1}}\tilde{x}_2 + G_{f_{k1}}\tilde{x}_3(t) + G_{p_{k1}}\tilde{x}_4(t) + G_{x_{k1}}\tilde{x}_5(t) \\ & + G_{I_{k2}}\tilde{x}_6(t) + G_{ACE_{k2}}\tilde{x}_7 + G_{f_{k2}}\tilde{x}_8(t) + G_{p_{k2}}\tilde{x}_9(t) + G_{x_{k2}}\tilde{x}_{10}(t) \end{aligned} \tag{4.21}$$

4.4 Main Results for Problem 3: The Team Optimal Control Method

4.4.1 Solution for Problem 3

In what follows, it is shown that the performance index (4.2) can be expressed with respect to the state-space formulations of Propositions 10 and 12.

Proposition 13. *Suppose Assumption 1 holds. For any* $(\mathbf{L}, G)$*,*

$$J_2^{HT}(L,G) := \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}^\mathsf{T} Q1\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}+R_1u_1^i(t)^2\right]\right.\right.$$
$$\left.\left.+\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}^\mathsf{T} Q2\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}+R_2u_2^i(t)^2\right]+\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\\ \tilde{x}_3(t)\\ \tilde{x}_4(t)\\ \tilde{x}_5(t)\\ \tilde{x}_6(t)\\ \tilde{x}_7(t)\\ \tilde{x}_8(t)\\ \tilde{x}_9(t)\\ \tilde{x}_{10}(t)\end{bmatrix}^\mathsf{T} P\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\\ \tilde{x}_3(t)\\ \tilde{x}_4(t)\\ \tilde{x}_5(t)\\ \tilde{x}_6(t)\\ \tilde{x}_7(t)\\ \tilde{x}_8(t)\\ \tilde{x}_9(t)\\ \tilde{x}_{10}(t)\end{bmatrix}\right)dt\right], \quad (4.22)$$

where

$$Q_1 := \begin{bmatrix}\beta_1^2 q_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, Q_2 := \begin{bmatrix}\beta_2^2 q_2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and

$$P := \begin{bmatrix} q_{I1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{I2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof.

$$\begin{aligned}
&\frac{1}{n_k}\sum_{i=1}^{n_k}[(ACE_k^i)^2 - (A\bar{C}E_k)^2] \\
&= \frac{1}{n_k}\sum_{i=1}^{n_k}[(\beta_k\Delta f_k^i + \frac{1}{n_k}\Delta P_{tiek})^2 - (\beta_k\Delta\bar{f}_k + \frac{1}{n_k}\Delta P_{tie_k})^2] \\
&= \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i)^2 - \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta\bar{f}_k)^2 + 2\beta_k\frac{1}{n_k}\sum_{i=1}^{n_k}\Delta f_k^i\frac{1}{n_k}\Delta P_{tie_k} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - 2\beta_k\frac{1}{n_k}\sum_{i=1}^{n_k}\Delta\bar{f}_k\frac{1}{n_k}\Delta P_{tie_k} \\
&= \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i)^2 - \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta\bar{f}_k)^2 \\
&= \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}((\Delta f_k^i - \Delta\bar{f}_k) + \Delta\bar{f}_k)^2 - \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta\bar{f}_k)^2 \\
&= \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i - \Delta\bar{f}_k)^2 + 2\beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i - \Delta\bar{f}_k)\Delta\bar{f}_k \\
&= \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i - \Delta\bar{f}_k)^2 \\
&\qquad\qquad\qquad\qquad \Downarrow \\
&\frac{1}{n_k}\sum_{i=1}^{n_k}[(ACE_k^i)^2 = \beta_k^2\frac{1}{n_k}\sum_{i=1}^{n_k}(\Delta f_k^i - \Delta\bar{f}_k)^2 + \frac{1}{n_k}\sum_{i=1}^{n_k}(A\bar{C}E_k)^2
\end{aligned} \tag{4.23}$$

Thus P is defined as in Proposition (4.22).

Also, the cost function (4.2) can be rewritten in the form of global variables listed in (4.22) □

In this paper, we impose the following assumption on the dynamics and cost.

Assumption 5. *For every t, Q_k, $\tilde{Q}$ and R_k are symmetric matrices that satisfy*

$$\begin{aligned} Q_k \geq 0, \forall k, \quad & \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \geq 0 \\ R_k > 0, \forall k, \quad & \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} > 0 \end{aligned} \tag{4.24}$$

In this part we assume $R_1 = R_2$

Assumption 6. *(A_k, B_k) and $(\tilde{A}, \tilde{B})$ are stabilizable and $(A_k, Q_k^{1/2})$ and $(\tilde{A}, \tilde{Q}^{1/2})$ are detectable.*

Let $z(t) := (\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t), \tilde{x}_4(t), \tilde{x}_5(t), \tilde{x}_6(t), \tilde{x}_7(t), \tilde{x}_8(t), \tilde{x}_9(t), \tilde{x}_{10}(t))$.
From Propositions 12,

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B} \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix} + \tilde{C} \tag{4.25}$$

Thus, for any feedback L, G

$$\begin{aligned} J_2^{HT}(L,G) = \lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} \Bigg[\frac{1}{n_1} \sum_{i=1}^{n_1} \Bigg(& \begin{bmatrix} \breve{x}_{11}^i(t) \\ \breve{x}_{12}^i(t) \\ \breve{x}_{13}^i(t) \end{bmatrix}^{\intercal} Q_1 \begin{bmatrix} \breve{x}_{11}^i(t) \\ \breve{x}_{12}^i(t) \\ \breve{x}_{13}^i(t) \end{bmatrix} + R\breve{u}_1^i(t)^2 \Bigg) \\ & + \frac{1}{n_2} \sum_{i=1}^{n_2} \Bigg(\begin{bmatrix} \breve{x}_{21}^i(t) \\ \breve{x}_{22}^i(t) \\ \breve{x}_{23}^i(t) \end{bmatrix}^{\intercal} Q_2 \begin{bmatrix} \breve{x}_{21}^i(t) \\ \breve{x}_{22}^i(t) \\ \breve{x}_{23}^i(t) \end{bmatrix} + R\breve{u}_2^i(t)^2 \Bigg) \Bigg] dt \\ + \lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} \Bigg(z(t)^{\intercal} P z(t) + & \begin{bmatrix} \bar{u}_1 & 0 \\ 0 & \bar{u}_2 \end{bmatrix}^{\intercal} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} \bar{u}_1 & 0 \\ 0 & \bar{u}_2 \end{bmatrix} \Bigg) dt \end{aligned} \tag{4.26}$$

To this end, we use the following result.

Theorem 3. *Under Assumptions 1, 5, 6, the optimal solution of Problem 3 is given as follows:*

For area 1:

$$\begin{aligned}
&{L^i_{f_1}}^* = \breve{K}_{11}, \quad {L^i_{p_1}}^* = \breve{K}_{12}, \quad {L^i_{x_1}}^* = \breve{K}_{13},\\
&{G_{I_{11}}}^* = \bar{K}_{11}, \quad {G_{ACE_{21}}}^* = \bar{K}_{12},\\
&{G_{f_{11}}}^* = \bar{K}_{13} - \breve{K}_{11}, \quad {G_{p_{11}}}^* = \bar{K}_{14} - \breve{K}_{12}, \quad {G_{x_{11}}}^* = \bar{K}_{15} - \breve{K}_{13},\\
&{G_{I_{11}}}^* = \bar{K}_{16}, \quad {G_{ACE_{12}}}^* = \bar{K}_{17},\\
&{G_{f_{12}}}^* = \bar{K}_{18}, \quad {G_{p_{12}}}^* = \bar{K}_{19}, \quad {G_{x_{12}}}^* = \bar{K}_{110}
\end{aligned} \tag{4.27}$$

For area 2:

$$\begin{aligned}
&{L^i_{f_2}}^* = \breve{K}_{21}, \quad {L^i_{p_2}}^* = \breve{K}_{22}, \quad {L^i_{x_2}}^* = \breve{K}_{23},\\
&{G_{I_{21}}}^* = \bar{K}_{21}, \quad {G_{ACE_{21}}}^* = \bar{K}_{22},\\
&{G_{f_{21}}}^* = \bar{K}_{23}, \quad {G_{p_{21}}}^* = \bar{K}_{24}, \quad {G_{x_{21}}}^* = \bar{K}_{25},\\
&{G_{I_{21}}}^* = \bar{K}_{26}, \quad {G_{ACE_{22}}}^* = \bar{K}_{27},\\
&{G_{f_{22}}}^* = \bar{K}_{28} - \breve{K}_{21}, \quad {G_{p_{22}}}^* = \bar{K}_{29} - \breve{K}_{22}, \quad {G_{x_{22}}}^* = \bar{K}_{210} - \breve{K}_{23}
\end{aligned} \tag{4.28}$$

where

$$\begin{aligned}
&[\breve{K}_{11}, \breve{K}_{12}, \breve{K}_{13}] := \text{Feedback}(A_1, B_1, Q_1, R_1),\\
&[\breve{K}_{21}, \breve{K}_{22}, \breve{K}_{23}] := \text{Feedback}(A_2, B_2, Q_2, R_2),\\
&\begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{K}_{14} & \bar{K}_{15} & \bar{K}_{16} & \bar{K}_{17} & \bar{K}_{18} & \bar{K}_{19} & \bar{K}_{110} \\ \bar{K}_{21} & \bar{K}_{22} & \bar{K}_{23} & \bar{K}_{24} & \bar{K}_{25} & \bar{K}_{26} & \bar{K}_{27} & \bar{K}_{28} & \bar{K}_{29} & \bar{K}_{210} \end{bmatrix}\\
&\qquad\qquad := \text{Feedback}(\tilde{A}, \tilde{B}, \tilde{P}, \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}).
\end{aligned} \tag{4.29}$$

Proof. Same as in Theorem 1, the optimal solution is given by:

$$\breve{u}^i_k(t) = \breve{K}_k \begin{bmatrix} \breve{x}^i_{k1}(t) \\ \breve{x}^i_{k2}(t) \\ \breve{x}^i_{k3}(t) \end{bmatrix}, \quad \bar{u}_k = \bar{K}_k z(t) \tag{4.30}$$

Then from the definition of $\breve{u}^i_k(t)$, $u^i_k(t) = \breve{u}^i_k(t) + \bar{u}_k(t)$, the optimal strategy of Problem 3 provided by (4.30) can be rewritten as follows:

$$\begin{aligned}
u_1^i(t) &= \breve{K}_{11}x_{11}^i(t) + \breve{K}_{12}x_{12}^i(t) + \breve{K}_{13}x_{13}^i(t) \\
&+ \bar{K}_{11}\tilde{x}_1(t) + \bar{K}_{12}\tilde{x}_2(t) \\
&+ (\bar{K}_{13} - \breve{K}_{11})\tilde{x}_3(t) + (\bar{K}_{14} - \breve{K}_{12})\tilde{x}_4(t) + (\bar{K}_{15} - \breve{K}_{13})\tilde{x}_5(t) \\
&+ \bar{K}_{16}\tilde{x}_6(t) + \bar{K}_{17}\tilde{x}_7(t) \\
&+ \bar{K}_{18}\tilde{x}_8(t) + \bar{K}_{19}\tilde{x}_9(t) + \bar{K}_{110}\tilde{x}_{10}(t) \\
u_2^i(t) &= \breve{K}_{21}x_{21}^i(t) + \breve{K}_{22}x_{22}^i(t) + \breve{K}_{23}x_{23}^i(t) \\
&+ \bar{K}_{21}\tilde{x}_1(t) + \bar{K}_{22}\tilde{x}_2(t) \\
&+ \bar{K}_{23}\tilde{x}_3(t) + \bar{K}_{24}\tilde{x}_4(t) + \bar{K}_{25}\tilde{x}_5(t) \\
&+ \bar{K}_{26}\tilde{x}_6(t) + \bar{K}_{27}\tilde{x}_7(t) \\
&+ \bar{(}\bar{K}_{28} - \breve{K}_{21})\tilde{x}_8(t) + (\bar{K}_{29} - \breve{K}_{22})\tilde{x}_9(t) + (\bar{K}_{210} - \breve{K}_{23})\tilde{x}_{10}(t)
\end{aligned} \tag{4.31}$$

Here (4.31), is in a form similar to (4.21). Therefore,it should also be optimal for Problem 3.

□

4.4.2 Numerical Results for Problem 3 : The Team Optimal Control Method

Consider 2 homogeneous populations of n generators, representing 2 different areas, where the distribution of initial difference of frequency in Area 1, Δf_1^i is uniform between 0 and 0.05 Hz, that of Area 2, Δf_2^i is uniform between -0.08 and 0 Hz (so that the total summation of frequency deviation does not exceed 0.15 p.u.Hz) and that of ΔP_g^i and ΔX_g^i are also uniform around 0 between about -0.02/n and 0.02/n p.u.MW (because the per unit base used here are the areawise base so that the actual per unit number will seems to be small).

We wish to achieve an objective that the average of each area goes to 0 thus all systems run in its nominal value with disturbance added only in area 2. Besides, to keep the same per unit base we assume that the 2 area shares the same system dynamics and same number of agents. So parameters used in the simulation are as follows:

$$q_{I1} = q_{I2} = 50, \quad q_1 = q_2 = 30, \quad R_1 = R_2 = 15 \tag{4.32}$$

$$
\begin{aligned}
&K_P = 120 \quad T_P = 20 \quad T_T = 0.3 \quad T_G = 0.08 \\
&T_{12} = 0.545 \quad a12 = -1 \quad \beta_1 = \beta_2 = 0.425
\end{aligned}
\tag{4.33}
$$

So that:

$$
G_g = \frac{1}{0.08s+1} \qquad G_t = \frac{1}{0.3s+1} \qquad G_p = \frac{120}{20s+1} \tag{4.34}
$$

100 machines starting from initial random values are considered in the simulation and the objective is to develop optimal team controllers such that mean frequency deviations and area control errors in the two areas asymptotically go to zero, following a series of load disturbances in Area 2. As shown in Figure 4.4, 20 randomly selected machine output trajectories are plotted, together with the mean of all 100, under conditions of 2 disturbances of size 0.03/n and 0.02/n p.u.MW added to area 2 at times $\frac{T}{4}$ and $\frac{T}{3}$, with $T = 50$. Besides, the frequency tracking errors in each of the areas, the values of ACE are also displayed.

The final global gain as follows:

```
g(area 1) =  1.8279   −0.3066    1.2358    1.3609    0.3135
             0.0269    0.2125   −0.1956   −0.1571   −0.0248
g(area 2) = −0.0240    0.0960   −0.1137   −0.1091   −0.0235
             1.8256   −0.8532    1.3922    1.8473    0.3553
```

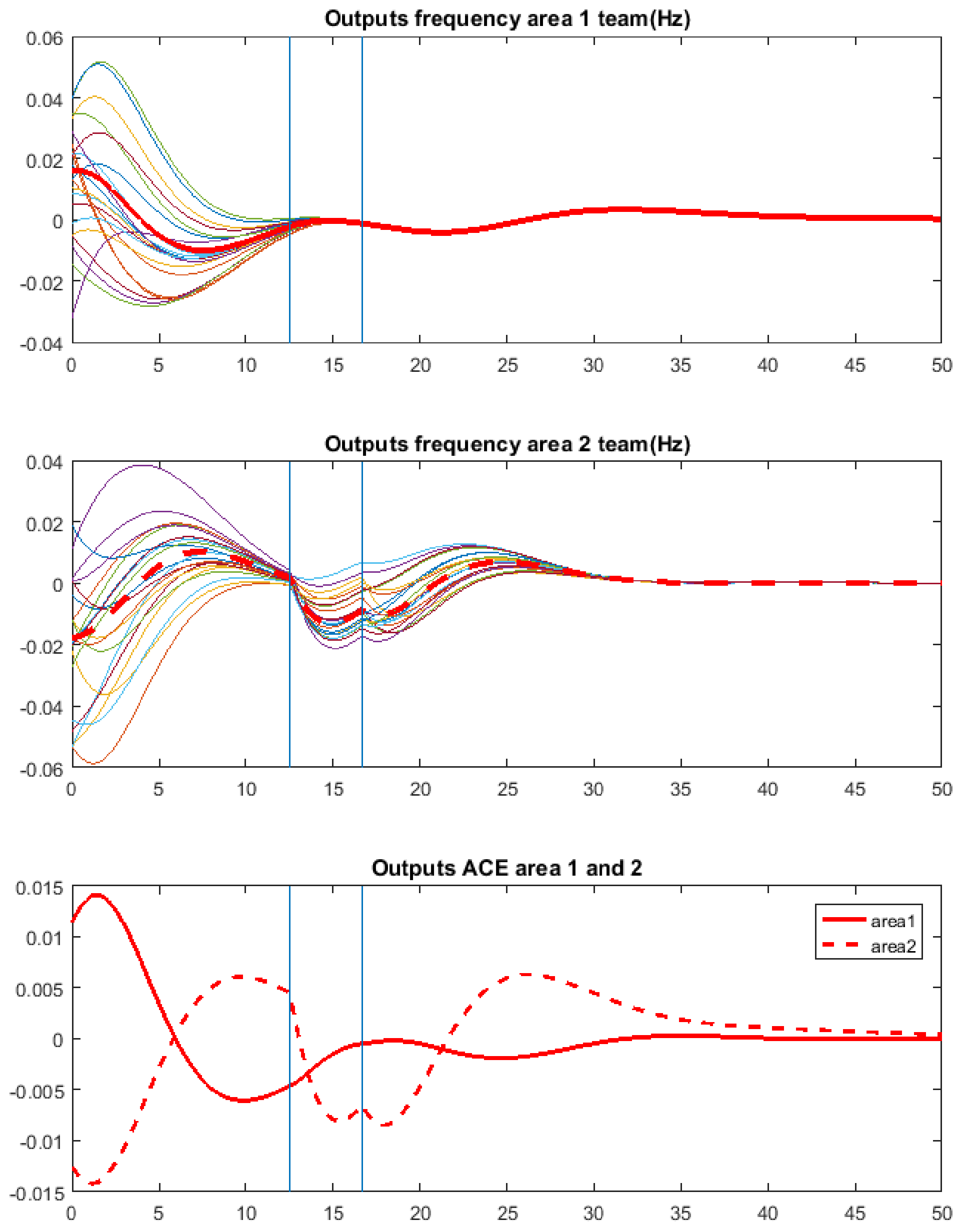

Figure 4.4 Top 2: Frequency deviation plots of 20 random selected agents out of 100 for the two area control system using team optimal solution under MFS-IS.Third: Area control signals for both areas. Disturbance instants correspond to vertical blue lines

In Figure 4.4, we observe the convergence of Δf_i's in both areas to zero and the self-regulation of Area 2 to keep stable when disturbances are added. Notice that Area 1 does provide help to Area 2 following disturbances. However, the definition of ACE is such that Area1 which is not subjected to disturbances of its own rapidly records an ACE nearing zero, unlike Area 2 which undergoes the disturbances. This confirms the usefulness of ACE in identifying the areas where generation setpoint changes are really required.

4.5 Main Results for Problem 4 : The Game Theoretic View.

4.5.1 Game Solution for Areawise Aggregated Model.

It is hard to combine game theory with team theory. We cannot simply carry the decomposition as in Section 4.4 because the actual system dynamics of each area will depend on that of the other area. To solve this problem, we try at first to find a game theoretic solution for a simpler aggregate areawise interconnected macromachine models of the two areas. The resulting feedback controls will correspond to the set of global feedback policies that we already encountered in the team optimal earlier formulation. Subsequently, each individual inherits the global feedback gains and computes team optimal feedback gains within its own area through solving independent Riccati equations..

Proposition 14. *The system dynamics of the aggregate model is the same as the global state space representation in the previous team model* (4.16)

Proof. The system dynamics of the global state in the previous model is linear so it will not be affected by multiplication. Furthermore, for simplicity, the two areas are assumed to have the same number of agents so that the values for each area are multiplied by the same number n, so that $n_1 = n_2 = n$ $\square$

We propose the corresponding cost functions for the aggregated macromachine models as follows:

$$\begin{aligned} \bar{J}_{21}^{HG}(G_1, G_2) &= \lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(z(t)^\intercal \tilde{P}_1 z(t) + R_1 \bar{u}_1(t)^2\right)dt\right] \\ \bar{J}_{22}^{HG}(G_1, G_2) &= \lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(z(t)^\intercal \tilde{P}_2 z(t) + R_2 \bar{u}_2(t)^2\right)dt\right] \end{aligned} \tag{4.35}$$

with $z(t) := (\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t), \tilde{x}_4(t), \tilde{x}_5(t), \tilde{x}_6(t), \tilde{x}_7(t), \tilde{x}_8(t), \tilde{x}_9(t), \tilde{x}_{10}(t))$.

and

$$P_1 := \begin{bmatrix} q_{I1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{I12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_2 := \begin{bmatrix} q_{I21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{I2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As we shall see, for the specific numerical case investigated, we observe that $\bar{J}_{21}^{HG}(G_1, G_2)$ and $\bar{J}_{22}^{HG}(G_1, G_2)$ are decoupled from each other. More specifically, in numerical experiments reported in Section 4.5.2, we find that the linear feedback laws associated to the global macromachine states for areas 1, 2 can be approximately expressed in terms of global state variables respectively attached to areas 1 and 2 because as it turns out, in each case, the gain parameters relating to the outside area are negligible.

Proposition 15. *$\bar{J}_{21}^{HG}(G_1, G_2), \bar{J}_{22}^{HG}(G_1, G_2)$ has the following two players game theoretic feedback solution:*

$$\bar{u}_1^* = -R_1^{-1} B_1^T \bar{K}_1 \tilde{X}, \quad \bar{u}_2^* = -R_2^{-1} B_1^T \bar{K}_2 \tilde{X} \tag{4.36}$$

provided there exists solutions $\bar{K}_1$ and $\bar{K}_2$ to the following system of coupled algebraic Riccati *equations:*

$$\begin{aligned}\dot{\bar{K}}_1 &= -\tilde{A}^T\bar{K}_1 - \bar{K}_1\tilde{A} - \tilde{P}_1 + \bar{K}_1 S_{11}\bar{K}_1 + K_1\bar{S}_{22}\bar{K}_2 + \bar{K}_2 S_{22}\bar{K}_1 \\ \dot{\bar{K}}_2 &= -\tilde{A}^T\bar{K}_2 - \bar{K}_2\tilde{A} - \tilde{P}_2 + \bar{K}_2 S_{22}\bar{K}_2 + \bar{K}_1 S_{11}\bar{K}_2 + \bar{K}_2 S_{11}\bar{K}_1\end{aligned} \tag{4.37}$$

with $S_{jj} = B_j R_j^{-1} B_j^T$

Proof. see [Basar and Olsder, 1999] □

It is rather difficult to compute solutions of coupled Riccati equations. Thus, we implement an algorithm (Algorithm 1), proposed and analysed in [Freiling et al., 1996].

Algorithm 1 Approaching Algorithm for Coupled Riccati equation

procedure
  $\bar{K}_1^0 \leftarrow care(\tilde{A}, B_1, \tilde{P}_1, R_1)$
  $\bar{K}_2^0 \leftarrow care(\tilde{A}, B_2, \tilde{P}_2, R_2)$
  while $\Delta K_1 > \epsilon$ or $\Delta K_2 > \epsilon$ **do**
    $0 = K_1^{c+1}(\tilde{A} - S_{22}K_2^c) + (\tilde{A} - S_{22}K_2^c)^T K_1^{c+1} + \tilde{P}_1 - K_1^{c+1} S_{11} K_1^{c+1}$
    $0 = K_2^{c+1}(\tilde{A} - S_{11}K_1^c) + (\tilde{A} - S_{11}K_1^c)^T K_2^{c+1} + \tilde{P}_2 - K_2^{c+1} S_{22} K_2^{c+1}$
    $\Delta k_1 = \|K_1^{c+1} - K_1^c\|$
    $\Delta k_2 = \|K_2^{c+1} - K_2^c\|$
    $i \leftarrow i + 1$
    if $i > 1000$ **then return** false, the coupled Riccati equation has no solution.
    $\bar{u}_1^* \leftarrow -R_1^{-1} B_1^T \bar{K}_1 \tilde{X}$
    $\bar{u}_2^* \leftarrow -R_2^{-1} B_1^T \bar{K}_2 \tilde{X}$

In numerical experiments performed on our two area systems which, as formulated, satisfies a necessary convergence condition mentioned in [Freiling et al., 1996] , we observe satisfactory convergence to a solution of (4.37).

4.5.2 Numerical Verification of Existence of Game Solution for Problem 4

Parameters are the same as in Section 4.4.2.

By numerical computation we find the global gains listed below and inducing in areas 1 and 2 the trajectories in red - respectively in full and dotted lines for the fully coupled solutions -, and blue - respectively in full and dotted lines-, for the areawise decoupled approximate solutions on Figure 4.5.

```
g1 =    1.8375   −0.3165    1.1962    1.3189    0.3050
        0.1429    0.2798   −0.1448   −0.0868   −0.0173
g2 =    0.1173    0.1838   −0.0623   −0.0407   −0.0121
        1.8353   −0.8468    1.3463    1.7902    0.3458
```

```
g1c =    1.8375   −0.3165    1.1962    1.3189    0.3050
          0         0          0         0          0
g2c =     0         0          0         0          0
         1.8353   −0.8468    1.3463    1.7902    0.3458
```

It can be observed in Figure 4.5 that the feedback strategies obtained for each area by neglecting the small feedback gains on macrostates outside the area of interest yield nearly the same behavior, thus validating current decoupled LFC approaches, at least for our particular numerical setting.

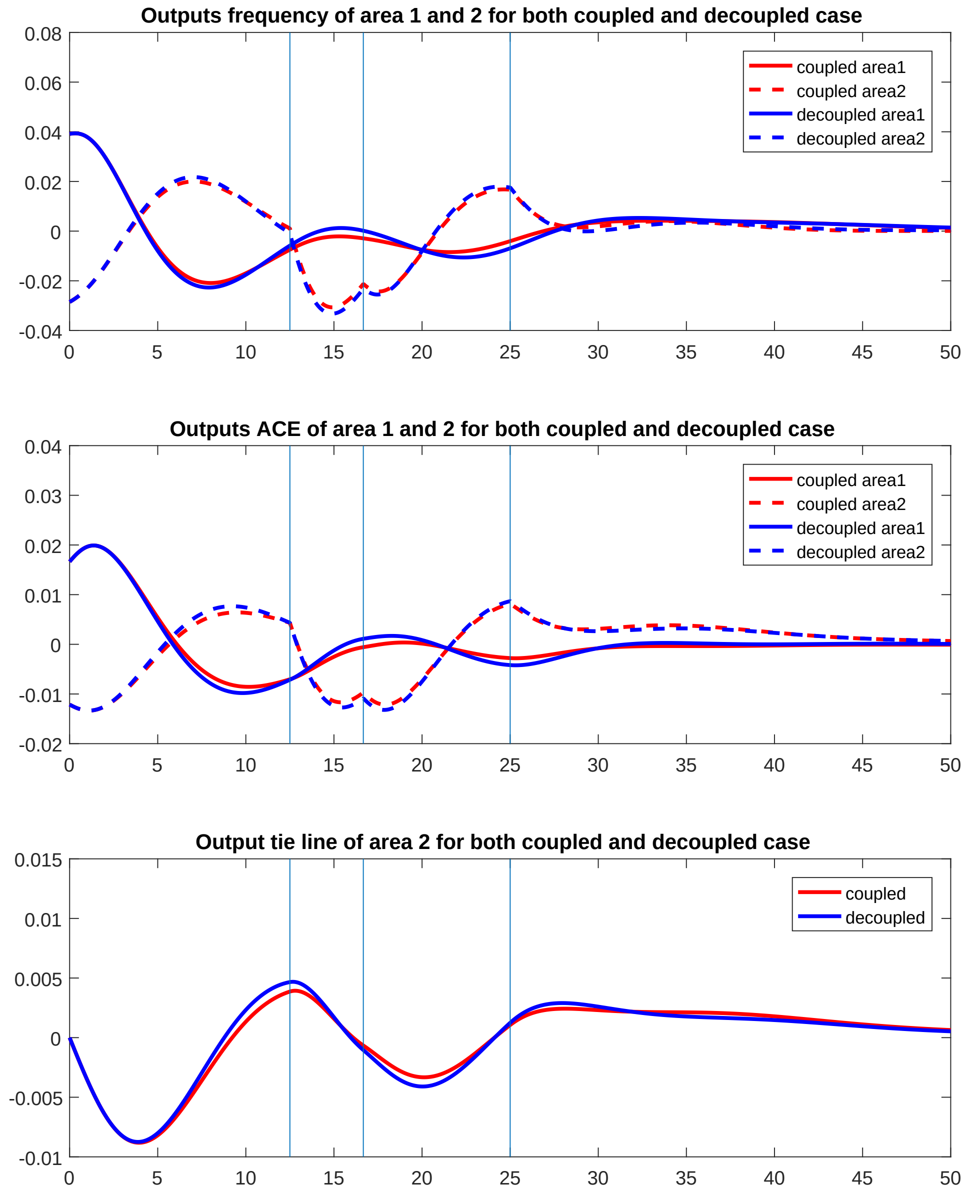

Figure 4.5 The plots of mean ACE and the tie line power exchange between the two areas for a team optimal control system under MFS-IS with both feedback on states from all areas, and decoupled areawise feedback structures.

4.5.3 Completion of Partial Game/Partial Team Solution for Problem 4

We inherit the decoupled global gains as obtained from the game theoretic solution computed for the simplified areawise interacting macromachines. We then compute in what follows the local individual machine control policies that would complete the hybrid partial game across areas/ partial team within area solution concept we are looking for.

Proposition 16. *The areawise cost functions for the partial team solution in each area with the global gain inherited from Section 4.5.1 are as follows:*

$$
\begin{aligned}
\breve{J}_{21}^{HG}(L_1) &= \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}^\intercal Q_1\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}+R\breve{u}_1^i(t)^2\right)dt\right]\\
\breve{J}_{22}^{HG}(L_2) &= \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_2}\sum_{i=1}^{n_2}\left(\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}^\intercal Q_1\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}+R\breve{u}_2^i(t)^2\right)dt\right]
\end{aligned}
\tag{4.38}
$$

Proof. Same as in proof of Proposition 12

$$
\begin{aligned}
&J_{21}^{HG}(L_1,G_1,G_2) :=\\
&\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(R\breve{u}_1^i(t)^2+R_1\bar{u}_1(t)^2\right.\right.\\
&\qquad\left.\left.+\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}^\intercal Q_1\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}+z(t)^\intercal P_1 z(t)\right)dt\right]\\
&=\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}^\intercal Q_1\begin{bmatrix}\breve{x}_{11}^i(t)\\ \breve{x}_{12}^i(t)\\ \breve{x}_{13}^i(t)\end{bmatrix}+R\breve{u}_1^i(t)^2\right)dt\right]\\
&\qquad+\lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(z(t)^\intercal\tilde{P}_1 z(t)+R_1\bar{u}_1(t)^2\right)dt\right]\\
&J_{22}^{HG}(L_2,G_1,G_2) :=\\
&\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_2}\sum_{i=1}^{n_2}\left(R\breve{u}_2^i(t)^2+R_2\bar{u}_2(t)^2\right.\right.\\
&\qquad\left.\left.+\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}^\intercal Q_2\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}+z(t)^\intercal P_2 z(t)\right)dt\right]\\
&=\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\left[\frac{1}{n_2}\sum_{i=1}^{n_2}\left(\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}^\intercal Q_2\begin{bmatrix}\breve{x}_{21}^i(t)\\ \breve{x}_{22}^i(t)\\ \breve{x}_{23}^i(t)\end{bmatrix}+R\breve{u}_2^i(t)^2\right)dt\right]\\
&\qquad+\lim_{T\to\infty}\left[\frac{1}{T}\int_{t=0}^{T}\left(z(t)^\intercal\tilde{P}_2 z(t)+R_2\bar{u}_2(t)^2\right)dt\right]
\end{aligned}
\tag{4.39}
$$

Thus

$$J_{21}^{HG}(L_1, G_1, G_2) = \breve{J}_{21}^{HG}(L_1) + \bar{J}_{21}^{HG}(G_1, G_2) \tag{4.40}$$
$$J_{22}^{HG}(L_2, G_1, G_2) = \breve{J}_{22}^{HG}(L_2) + \bar{J}_{22}^{HG}(G_1, G_2) \tag{4.41}$$

with $\bar{J}_{21}^{HG}, \bar{J}_{22}^{HG}$ predefined in Section 4.5.1. □

Each individual inherits the solution of the interarea game with global costs $\bar{J}_{21}^{HG}(G_1, G_2)$ and $\bar{J}_{22}^{HG}(G_1, G_2)$ inducing the global feedback gains in each area, independent of the other area.Then each individual obtains the solution of $\breve{J}_{21}^{HG}(L_1)$ and $\breve{J}_{22}^{HG}(L_2)$, as local gain which controls the discrepancy between an individual's states and the mean field states as in the earlier team optimal approach :

$$\breve{u}_k^i(t) = -R_k^{-1} B_k^T \breve{K}_k \begin{bmatrix} \breve{x}_{k1}^i(t) \\ \breve{x}_{k2}^i(t) \\ \breve{x}_{k3}^i(t) \end{bmatrix} \tag{4.42}$$

with optimal gains, as in Proposition 11, given by :

$$\breve{K}_1 = \text{Feedback}(A_1, B_1, Q_1, R_1), \quad \breve{K}_2 = \text{Feedback}(A_2, B_2, Q_2, R_2), \tag{4.43}$$

Theorem 4. *Under Assumptions 1 and 6, the partial Nash/partial team equilibrium solution of Problem 4 is given as follows:*

For area 1:

$$\begin{aligned}
&{L_{f_1}^i}^* = \breve{K}_{11}, \quad {L_{p_1}^i}^* = \breve{K}_{12}, \quad {L_{x_1}^i}^* = \breve{K}_{13}, \\
&{G_{I_{11}}}^* = \bar{K}_{11}, \quad {G_{ACE_{21}}}^* = \bar{K}_{12}, \\
&{G_{f_{11}}}^* = \bar{K}_{13} - \breve{K}_{11}, \quad {G_{p_{11}}}^* = \bar{K}_{14} - \breve{K}_{12}, \quad {G_{x_{11}}}^* = \bar{K}_{15} - \breve{K}_{13}, \\
&{G_{I_{11}}}^* = 0, \quad {G_{ACE_{12}}}^* = 0, \\
&{G_{f_{12}}}^* = 0, \quad {G_{p_{12}}}^* = 0, \quad {G_{x_{12}}}^* = 0
\end{aligned} \tag{4.44}$$

For area 2:

$$
\begin{aligned}
&{L^i_{f_2}}^* = \breve{K}_{21}, \quad {L^i_{p_2}}^* = \breve{K}_{22}, \quad {L^i_{x_2}}^* = \breve{K}_{23}, \\
&{G_{I_{21}}}^* = 0, \quad {G_{ACE_{21}}}^* = 0, \\
&{G_{f_{21}}}^* = 0, \quad {G_{p_{21}}}^* = 0, \quad {G_{x_{21}}}^* = 0, \\
&{G_{I_{21}}}^* = \bar{K}_{26}, \quad {G_{ACE_{22}}}^* = \bar{K}_{27}, \\
&{G_{f_{22}}}^* = \bar{K}_{28} - \breve{K}_{21}, \quad {G_{p_{22}}}^* = \bar{K}_{29} - \breve{K}_{22}, \quad {G_{x_{22}}}^* = \bar{K}_{210} - \breve{K}_{23}
\end{aligned}
\tag{4.45}
$$

where

$$
\begin{aligned}
&[\breve{K}_{11}, \breve{K}_{12}, \breve{K}_{13}] := \text{Feedback}(A_1, B_1, Q_1, R_1), \\
&[\breve{K}_{21}, \breve{K}_{22}, \breve{K}_{23}] := \text{Feedback}(A_2, B_2, Q_2, R_2), \\
&\begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{K}_{14} & \bar{K}_{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{K}_{26} & \bar{K}_{27} & \bar{K}_{28} & \bar{K}_{29} & \bar{K}_{210} \end{bmatrix} := \begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \end{bmatrix}.
\end{aligned}
\tag{4.46}
$$

where $\bar{K}_1$, $\bar{K}_2$ *in* (4.37)

Proof. Same as in Theorem 3, from the definition of $\breve{x}^i_1(t)$ and $\breve{x}^i_2(t)$, the optimal strategy can be rewritten as follows:

$$
\begin{aligned}
u^i_1(t) = {} & \breve{K}_{11} x^i_{11}(t) + \breve{K}_{12} x^i_{12}(t) + \breve{K}_{13} x^i_{13}(t) \\
& + \bar{K}_{11} \tilde{x}_1(t) + \bar{K}_{12} \tilde{x}_2(t) \\
& + (\bar{K}_{13} - \breve{K}_{11}) \tilde{x}_3(t) + (\bar{K}_{14} - \breve{K}_{12}) \tilde{x}_4(t) + (\bar{K}_{15} - \breve{K}_{13}) \tilde{x}_5(t) \\
u^i_2(t) = {} & \breve{K}_{21} x^i_{21}(t) + \breve{K}_{22} x^i_{22}(t) + \breve{K}_{23} x^i_{23}(t) \\
& + \bar{K}_{26} \tilde{x}_6(t) + \bar{K}_{27} \tilde{x}_7(t) \\
& + \bar{(}\bar{K}_{28} - \breve{K}_{21}) \tilde{x}_8(t) + (\bar{K}_{29} - \breve{K}_{22}) \tilde{x}_9(t) + (\bar{K}_{210} - \breve{K}_{23}) \tilde{x}_{10}(t)
\end{aligned}
\tag{4.47}
$$

Here (4.47) is in a similar form of (4.21) Therefore, it should also be optimal for Problem 4. □

4.5.4 Numerical Results for Problem 4: The Game Theoretic Vie

Parameters are the same as in Section 4.4.2 only with external values $q_{I21} = q_{I12} = 5, q_{21} = q_{12} = 5$ and the result is as follows:

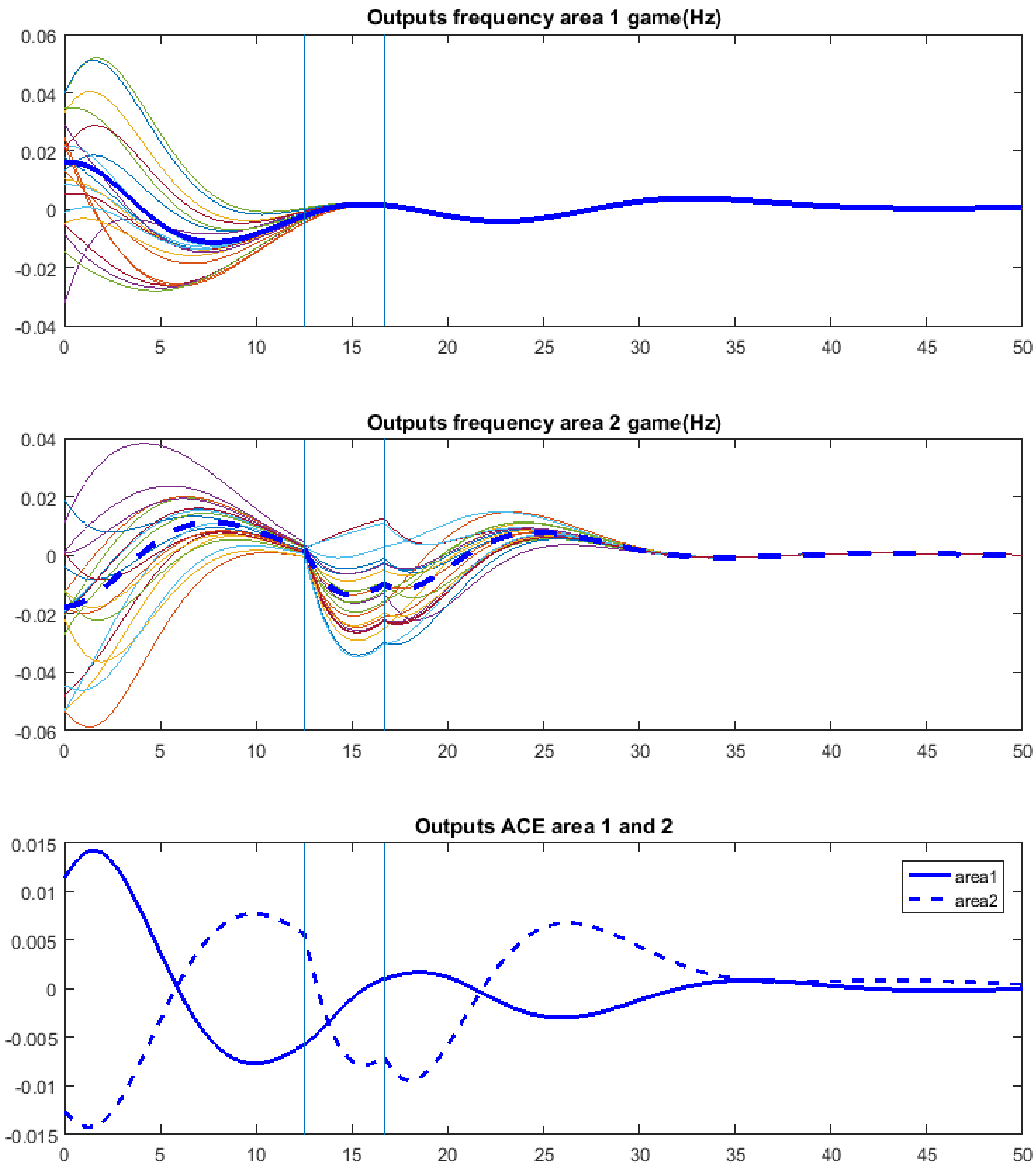

Figure 4.6 Top 2: Frequency deviation plots of 20 random selected agents out of 100 for the two area control system using nash optimal solution under MFS-IS. Third: Area control signals for both areas. Disturbance instants correspond to vertical blue lines.

100 systems starting from initial random values are considered in the simulation and the objective is to develop partial game/partial team based controllers such that mean frequency deviations and area control errors in the two areas asymptotically go to zero, following a series

of load disturbances in Area 2. As shown in Figure 4.6, 20 randomly selected machine output trajectories are plotted, together with the mean of all 100, under conditions of 3 disturbances of size 0.02 p.u.MW added to area 2 at times $\frac{T}{4}$, $\frac{T}{3}$ and $\frac{T}{2}$, with $T = 50$. Besides, the frequency tracking errors in each of the areas, the values of ACE are also displayed.

In Figure 4.6, we observe the convergence of Δf_i's in both areas to zero and the self-regulation of Area 2 to remain stable when disturbances are added. Notice that we could observe similar behaviours in the fully team optimal solution in Section 4.4.2. It is thus useful to compare the resulting dynamic behaviours. The results of this comparison are reported in Section 4.6.

4.6 Numerical Comparison of 2 Area LFC Problem for Both Solutions

Parameters are nearly the same as in Section 4.4.2 with only a slight change to the initial condition, where the frequencies of Area 1, Δf_1^i is uniform between 0.02 and 0.08 Hz while that of Area 2, Δf_2^i is uniform between -0.08 and -0.02 Hz.

Here 2 different methods, team optimal solution in Section 4.4, marked red, game solution in Section 4.5, marked blue, are shown in both 2 Figures. Besides, full line represent Area 1, which is not disturbed, while dotted line represent area 2, where 2 disturbances of $0.03/n$ and 0.02/n p.u.MW are added at $\frac{T}{4}$ and $\frac{T}{3}$, with $T = 50$.

Difference between these 2 solutions are listed in Table 4.3.

Table 4.3 Comparison of 2 solutions of Problem 3 and Problem 4

Situation	team optimum	Nash optimum
color	red	blue
variation of Δf	smaller	slightly bigger
ACE	smaller	slightly bigger
P_{tie}	smaller variation	bigger variation

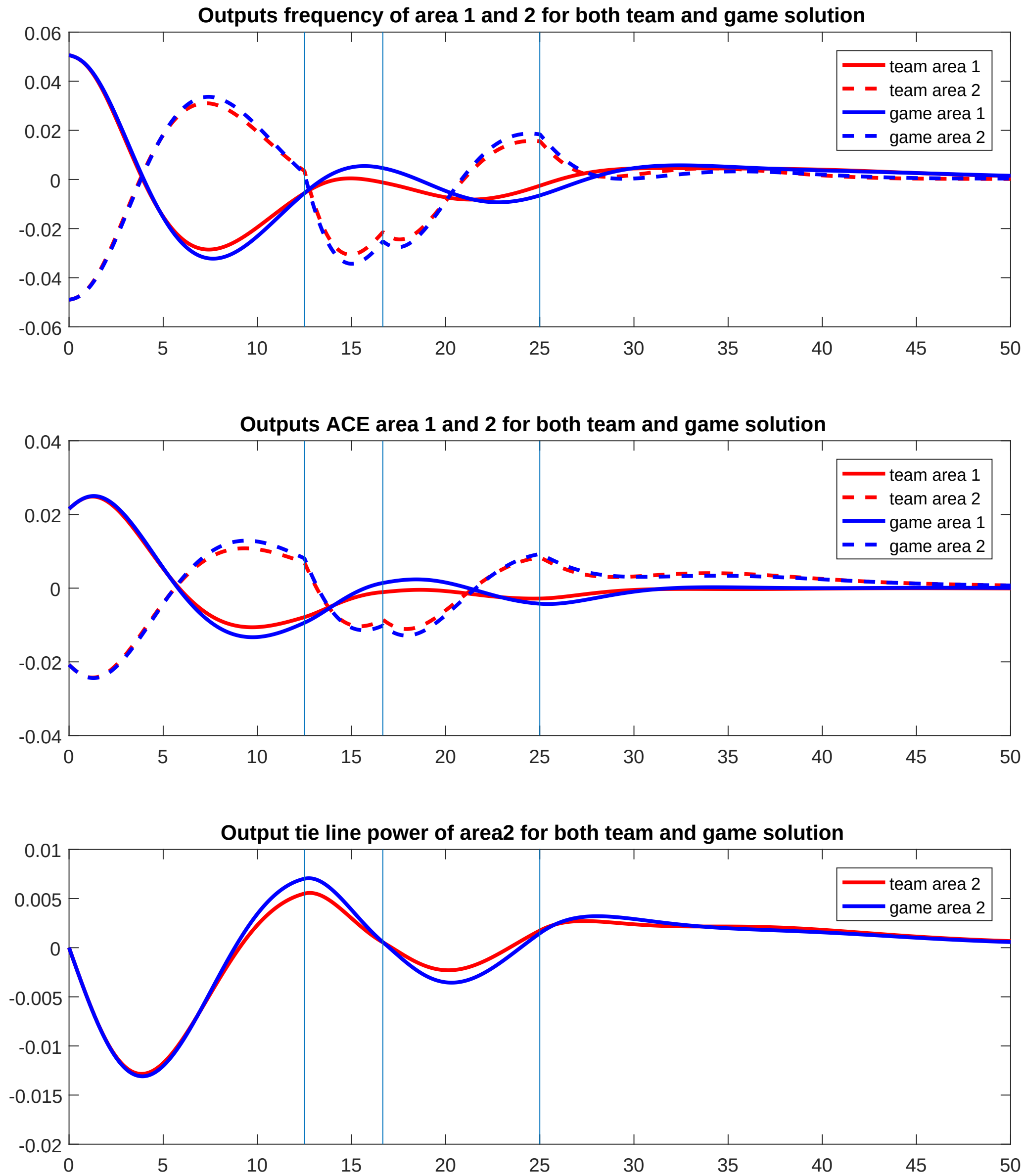

Figure 4.7 Plots of mean area control errors ACE and tie line power interchange for the two areas control systems under MFS-IS for both fully team optimal controls and partial team/partial game induced controls.

It can be observed in Figure 4.7 though the frequency deviation of both areas that the feedback strategies obtained for partial team/partial game solution yield a slightly more selfish behaviour on the part Area 1 which is not touched by the load disturbances even with the decoupling effect of the choice of ACE.

CHAPTER 5 CONCLUSION AND RESEARCH PERSPECTIVES

In this thesis, our goal has to been to revisit the important power systems automatic generation control, also known as the load frequency control problem, while hanging on to the precious engineering intuitions accumulated over time on this problem, yet relying on the contributions of linear quadratic optimal control and the more recent theory of linear quadratic mean field teams.

Traditionally, load frequency control has been carried out using classical proportional-integral control design and relying on an areawise measurable signal, the area control error or ACE. In the thesis, we have maintained the philosophy of using the area control error signal as a useful indicator of mismatch between generation and load within one area, as well as the idea that integral control could provide an automated way to adjust generation against piecewise constant slowly changing load disturbances; only we cast the control of ACE in either, (i) a purely optimal control framework within one area, as well as two interconnected areas which must collaborate together if they are under the responsibility of the same operator, or (ii) a more areawise self interested framework for interconnected areas operated by independent power companies and for which we formulate a dynamic game.

Our fundamental intuition has been that the current mode of control within a single area wherein the ACE signal reduces to "mean area frequency", is somewhat consistent with the results that would be obtained using team theory, i.e. cooperative control of multi-agent systems sharing the same cost function they need to collectively optimize, but having access to different information sets. Linear quadratic mean field team theory has the elegant characteristic that it relies on a *decomposition* of the team optimal control policy of multi-agent systems into an *agentwise local state linear feedback control*, to which a system mean state or *global state linear feedback* is added. As a consequence, a team optimal control policy relies only on a locally observed set of states, and the sharing of average quantities which are individual agent state averages over each subgroup of dynamically homogeneous machines. This is rather reminiscent of the current LFC structure which involves proportional local state feedback via locally based synchronous machine droop control, and a global integral control feedback via system frequency mean deviation, or more generally ACE which is also a global quantity. What mean field team theory teaches us in this case is that it may be useful to break up the global feedback control beyond a single average quantity into components which rely on multiple state average quantities within homogeneous subgroups of generators in the area. Of course, this may come at the cost of a more sophisticated measurement structure. This was studied in the first part of the thesis.

In the second part of the thesis, we have considered the case of two interconnected areas either cooperating in a team optimal consistent fashion, or competing across areas, yet collaborating within a single area, i.e. under a hybrid partial game/partial team organization of the control. In the first case, we had to reformulate the two area problem as one involving a single area with two non homogeneous subgroups of machines interacting via a tie line power signal which had to be construed as a "global" quantity. The result was feedback control policies depending on agentwise local states, but also on all global quantities including both ACE (as traditionally used), and the average of states within each area, i.e. the two areas would need to share their areawise global quantities. From an operational point of view, while possible, this may be impractical. Thus we verified what happens if one imposes areas decoupling by forcing the feedback gains involving global states belonging to an area other than that of interest to zero. At least, for our numerical setting, we could verify that the forcibly decoupled controllers performed nearly as well as the fully coupled ones.This of course is a vindication of the decoupling effect of the choice of ACE as a signal of mismatch between generation and load within one area. In the second case though, we had to apply game theory between two potentially competing areas. In order to keep computations at a reasonable level and in keeping with the idea that machines in a the same area will collaborate, we replace all machines in one area by a single macromachine. Thus the game is defined between two interconnected macromachines , and again in keeping with the area control error philosophy, ACE_1 and ACE_2 are each associated with a quadratic cost in their respective areas. If the corresponding linear quadratic game has a solution, it is considered as yielding the global part of an eventual areawise team optimal controller. However the Nash equilibrium of the dynamic game leads to linear feedback policies on global states of both areas, and this may not be practical. Again here, in each area, we set to zero the gains involving global variables specific to the other area, and we do note that this forcible decoupling does not significantly alter the resulting dynamics, at least in our numerical setting. We then proceed to compute the strictly local components of the control in a manner similar to that used to determine team optimal controllers. The result is a hybrid control, part Nash, and part team optimal. It is compared to the fully team optimal two area controller and one can observe a more selfish behavior on the part area 1 which is not touched by the load disturbances.
Our analysis indicates directions in which the current LFC schemes in power systems could be improved. Also, a mathematical setting sufficiently general has been created so as one

could explore cost functions which while still based on the ACE notion, could involve in a more specific fashion areawise local frequency deviations. This way, the extent to which one area could help another one in case of need could be controlled in a more systematic fashion. Finally, it would be of interest to analyze the case of two competing areas internally involving several distinct subgroups of homogeneous machines, as well as the case of multiple interconnected competing areas.

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APPENDIX A LINEAR QUADRATIC MEAN-FIELD TEAM

Consider a population of N agents that are partitioned into K disjoint sub-populations N_k, $k \in K := \{1..K\}$, such that all the notations are listed below:

Notation used for agent $i \in \mathcal{N}^k$ belonging to sub-population $k \in \mathcal{K}$	
$x_t^i \in \mathbb{R}^{d_x^k}$	State of agent i
$u_t^i \in \mathbb{R}^{d_u^k}$	Action of agent i
Notation used for sup-population $k \in \mathcal{K} = \{1, \ldots, K\}$	
$\mathcal{N}^k$	Entire sub-population k
$\bar{x}_t^k = \langle (x_t^i)_{i \in \mathcal{N}^k} \rangle$	Mean-field of states at time t
$\bar{u}_t^k = \langle (u_t^i)_{i \in \mathcal{N}^k} \rangle$	Mean-field of actions at time t
Notation used for entire population	
$\mathcal{N} = \bigcup_{k \in \mathcal{K}} \mathcal{N}^k$	Entire population
$\mathbf{x}_t = (x_t^i)_{i \in \mathcal{N}}$	Joint state of entire population at time t
$\mathbf{u}_t = (u_t^i)_{i \in \mathcal{N}}$	Joint action of entire population at time t
$\bar{\mathbf{x}}_t = \mathrm{vec}(\bar{x}_t^1, \ldots, \bar{x}_t^K)$	Mean-field of states of entire population at time t
$\bar{\mathbf{u}}_t = \mathrm{vec}(\bar{u}_t^1, \ldots, \bar{u}_t^K)$	Mean-field of actions of entire population at time t

According to [Arabneydi, 2016], the system dynamic of agent $i \in N^k$ of sub-population $k \in K$ is given by

$$x_{t+1}^i = A_t^k x_t^i + B_t^k u_t^i + D_t^k \bar{\mathbf{x}}_t + E_t^k \bar{\mathbf{u}}_t + w_t^i \tag{A.1}$$

Where $w_t^i \in W^k$ is the disturbance noise process.
The per-step cost at time $t \in \{1, ...T-1\}$ is given by

$$c_t(\mathbf{x}_t, \mathbf{u}_t, \bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t) = \bar{\mathbf{x}}_t^T P_t^x \bar{\mathbf{x}}_t + \bar{\mathbf{u}}_t^T P_t^u \bar{\mathbf{u}}_t + \sum_{k \in K} \sum_{i \in N^k} \frac{1}{N^k} [(x_t^i)^T Q_t^k x_t^i + (u_t^i)^T R_t^k u_t^i] \tag{A.2}$$

and the per-step cost at time $t = T$ is given by

$$c_T(\mathbf{x}_T, \bar{\mathbf{x}}_T) = \bar{\mathbf{x}}_t^T P_t^x \bar{\mathbf{x}}_t + \sum_{k \in K} \sum_{i \in N^k} \frac{1}{N^k} (x_T^i)^T Q_T^k x_T^i \tag{A.3}$$

The performance of strategy g is given by

$$J(g) = \mathbb{E}^g \left[\sum_{t=1}^{T-1} c_t(\mathbf{x}_t, \mathbf{u}_t, \bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t) + c_T(\mathbf{x}_T, \bar{\mathbf{x}}_T) \right] \tag{A.4}$$

where the expectation is with respect to the measure induced on all the system variables by the choice of strategy g.

The optimization problem is to find a strategy g that

$$J^* = J(g^*) = \inf_g J(g) \tag{A.5}$$

The model is built with the following assumptions:

Assumption 7. *The primitive random variables* $\{x_1; \{w_t\}_{t=1}^T\}$ *have zero mean and are mutually independent.*

Assumption 8. *For every* t*,* Q_k*,* $\tilde{Q}$ *and* R_k *are symmetric matrices that satisfy*

$$\begin{aligned} Q_k \geq 0, \forall k, diag(Q_t^1...Q_t^K) + P_t^x \geq 0 \\ R_k > 0, \forall k, diag(R_t^1...R_t^K) + P_t^u > 0 \end{aligned} \tag{A.6}$$

So that the main result is as follows:

1.Structure of optimal strategy:

The optimal strategy is unique and is linear in the local state and the mean-field of the system. In particular,

$$u_t^i = \breve{L}_t^k(x_t^i - \bar{x}_t^k) + \bar{L}_t^k \bar{\mathbf{x}}_t^k \tag{A.7}$$

2. Riccati equations:

Let

$$\begin{aligned} \bar{A}_t &= diag(A_t^1, ...A_t^K) + rows(D_t^1, ...D_t^K) \\ \bar{B}_t &= diag(B_t^1, ...B_t^K) + rows(E_t^1, ...E_t^K) \\ \bar{Q}_t &= diag(Q_t^1, ...Q_t^K) \quad \bar{R}_t = diag(R_t^1, ...R_t^K) \end{aligned} \tag{A.8}$$

Then, for $t \in \{1, ...T-1\}$:

$$\begin{aligned} \breve{L}_t^k &= -\,((B_t^k)^T \breve{M}_{t+1}^k B_t^k + R_t^k)^{-1}(B_t^k)\breve{M}_{t+1}^k A_t^k \\ \bar{L}_t^k &= -\,(\bar{B}_t)^T \bar{M}_{t+1}^k \bar{B}_t) + \bar{R}_t)^{-1} \bar{B}_t \bar{M}_{t+1} \bar{A}_t \end{aligned} \tag{A.9}$$

where $\breve{M}_{1:T}^k$ and $\bar{M}_{1:T}$ are the solutions of following Riccati equations:

$$\begin{aligned} \breve{M}_{1:T}^k &= DRE_T(A_{1:T}^k, B_{1:T}^k, Q_{1:T}^k, R_{1:T}^k) \\ \bar{M}_{1:T} &= DRE_T(\bar{A}_{1:T}, \bar{B}_{1:T}, \bar{Q}_{1:T} + P_{1:T}^x, \bar{R}_{1:T} + P_{1:T}^u) \end{aligned} \tag{A.10}$$